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Non-Archimedean Integration and Strict Topologies

A. K. KATSARAS Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece

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Introduction

Let $C_b(X,E)$ be the space of all bounded continuous functions from a zero-dimensional Hausdorff topological space X to a non-Archimedean Hausdorff locally convex space E. By $C_{rc}(X,E)$ we denote the space of all $f \in C_b(X,E)$ for which $f(X)$ is a relatively compact subset of E. In section 2 of this paper we show that, if E is polar and complete and Y a closed subset of X which is either compact or X is ultranormal, then there exists a linear map $T : C_{rc}(Y,E) \to C_{rc}(X,E)$ such that Tf is an extension of f and $\|Tf\|_p = \|f\|_p$ for all $f \in C_{rc}(Y,E)$ and every polar continuous seminorm p on E. Using this we identify in section 3 the completion of the space $C_b(X,E)$ under the strict topology β_o when E is polar. If $K(X)$ is the algebra of all clopen (i.e. both closed and open) subsets of X, we define in section 4 the product of certain $\mathbb{K}$-valued finitely-additive measures on $K(X)$ with E'-valued measures on $K(Y)$, where Y is another zero-dimensional topological space. Finally in sections 5 and 6 we define the so called (VR)-integral and Q-integral of functions in E^X with respect to certain measures on $K(X)$.

1 Preliminaries

Throughout this paper, $\mathbb{K}$ stands for a complete non-Archimedean valued field whose valuation is non-trivial. By a seminorm, on a vector space E over $\mathbb{K}$, we mean a non-Archimedean seminorm. Similarly, by a locally convex space we mean a non-Archimedean locally convex space over $\mathbb{K}$. For E a locally convex space, we denote by $cs(E)$ the collection of all continuous seminorms on E, by E' its dual space and by $\hat{E}$ its completion. If F is another locally convex space, then $E \otimes F$ will be the tensor product of E, F with the projective topology.

Let now X be a zero-dimensional Hausdorff topological space and E a Hausdorff locally convex space. We will denote by $\beta_o X$ the Banaschewski compactification of X (see [4]) and by $\upsilon_o X$ the $\mathbb{N}$-repletion of X ($\mathbb{N}$ is the set of natural numbers), i.e. the subspace of $\beta_o X$ consisting of all $x \in \beta_o X$ with the following property: For each sequence (V_n) of neighborhoods of x in $\beta_o X$ we have that $\bigcap V_n \cap X \neq \emptyset$. The space X is called $\mathbb{N}$-replete if $X = v_o X$. We will denote by $C_b(X, E)$ the space of all bounded continuous E-valued functions on X and by $C_{rc}(X, E)$ the space of all $f \in C_b(X, E)$ for which $f(X)$ is relatively compact in E. In case $E = \mathbb{K}$, we will simply write $C_b(X)$ and $C_{rc}(X)$ respectively. For $A \subset X$, we denote by χ_A the $\mathbb{K}$-valued characteristic function of A in X and by $\overline{A}^{\beta_o X}$ the closure of A in $\beta_o X$. Every $f \in C_{rc}(X, E)$ has a unique continuous extension f^{β_o} to all of $\beta_o X$. For f an E-valued function on X, p a seminorm on E and $A \subset X$, we define

$$\|f\|_p = \sup_{x \in X} p(f(x)), \quad \|f\|_{A,p} = \sup_{x \in A} p(f(x)).$$

The strict topology β_o on $C_b(X, E)$ (see [7]) is the locally convex topology generated by the seminorms $f \mapsto \|hf\|_p$, where $p \in cs(E)$ and h is in the space $B_o(X)$ of all bounded $\mathbb{K}$-valued functions on X which vanish at infinity, i.e. for each $\epsilon > 0$ there exists a compact subset Y of X such that $|h(x)| < \epsilon$ if x is not in Y. Let Ω be the family of all compact subsets of $\beta_o X$ which are disjoint from X. For $H \in \Omega$, let C_H be the space of all $h \in C_{rc}(X)$ whose continuous extension h^{β_o} vanishes on H. For $p \in cs(E)$, let $\beta_{H,p}$ be the locally convex topology on $C_b(X, E)$ generated by the seminorms $f \mapsto \|hf\|_p, h \in C_H$. The inductive limit of the topologies $\beta_{H,p}$, as H ranges over Ω, is denoted by β_p while β is the projective limit of the topologies $\beta_p, p \in cs(E)$. The following Theorem is proved in [11].

Theorem 1.1 *An absolutely convex subset V of $C_b(X, E)$ is a $\beta_{H,p}$-neighborhood of zero iff the following condition is satisfied: For each $r > 0$, there exist $\epsilon > 0$ and a clopen subset A of X, with $\bar{A}^{\beta_o X} \cap H = \emptyset$, such that*

$$\{f \in C_b(X, E) : \|f\|_p \leq r, \|f\|_{A,p} \leq \epsilon\} \subset V.$$

Let now $K(X)$ be the algebra of all clopen, (i.e. closed and open) subsets of X. We denote by $M(X, E')$ (see [6]) the space of all finitely-additive E'-valued measures m on $K(X)$ for which $m(K(X))$ is an equicontinuous subset of E'. For each m in $M(X, E')$ there exists $p \in cs(E)$ with $m_p(X) < \infty$, where, for $A \in K(X)$,

$$m_p(A) = \sup\{|m(B)s|/p(s) : p(s) \neq 0, A \supset B \in K(X)\}.$$

The space of all $m \in M(X, E')$ with $m_p(X) < \infty$ is denoted by $M_p(X, E')$. We denote by $M_\tau(X, E')$ the space of all $m \in M(X, E')$ such that, for every decreasing net (A_δ) of clopen subsets of X, with $\cap A_\delta = \emptyset$, there exists $p \in cs(E)$ such that $m_p(A_\delta) \to 0$. Also by $\mathcal{M}_{\tau,p}(X, E')$ we denote the space of all $m \in M_p(X, E')$ such that $m_p(A_\delta) \to 0$ for every decreasing net (A_δ) of clopen subsets of X with $\cap A_\delta = \emptyset$. Let

$$\mathcal{M}_\tau(X, E') = \bigcup_{p \in cs(E)} \mathcal{M}_{\tau,p}(X, E').$$

For $p \in cs(E)$, we denote by $M_{t,p}(X, E')$ the space of all $m \in M_p(X, E')$ for which m_p is tight, i.e. for every $\epsilon > 0$, there exists a compact subset Y of X such that

$m_p(A) \leq \epsilon$ if A is disjoint from Y. We define

$$M_t(X,E') = \bigcup_{p \in cs(E)} M_{t,p}(X,E').$$

As it is shown in [11], $\mathcal{M}_{\tau,p}(X,E') = M_{t,p}(X,E')$. In case $E = \mathbb{K}$, we write $M(X), M_\tau(X)$ and $M_t(X)$ for $M(X,E'), M_\tau(X,E')$ and $M_t(X,E')$, respectively. Also, for $\mu \in M(X)$, we define $|\mu|(A) = \mu_p(A)$, where $p = |.|$ is the valuation of $\mathbb{K}$.

Next, we recall the definition of the integral of an E-valued function f on X with respect to an $m \in M(X,E')$. For $A \in K(X), A \neq \emptyset$, let $\mathcal{D}_A$ denote the family of all $\alpha = \{A_1, \ldots, A_n : x_1, \ldots, x_n\}$, where $\{A_1, \ldots, A_n\}$ is a clopen partition of A and $x_i \in A_i$. We make $\mathcal{D}_A$ a directed set by defining $\alpha_1 \geq \alpha_2$ iff the partition of A in α_1 is a refinement of the one in α_2. For $f \in E^X, m \in M(X,E')$ and $\alpha = \{A_1, \ldots, A_n : x_1, \ldots, x_n\}$, we define $\omega_\alpha(f,m) = \sum_{i=1}^n m(A_i)f(x_i)$. If the $\lim_\alpha \omega_\alpha(f,m)$ exists in $\mathbb{K}$, we will say that f is m-integrable over A and denote this limit by $\int_A f dm$. We define the integral over the empty set to be 0. For $A = X$, we write simply $\int f dm$. It is easy to see that if f is m-integrable over X, then it is m-integrable over every $A \in K(X)$ and $\int_A f dm = \int \chi_A f dm$. Every $m \in M(X,E')$ defines a τ_u-continuous linear functional on $C_{rc}(X,E)$ by $f \mapsto \int f dm$ (see [6]). Also every $\phi \in (C_{rc}(X,E), \tau_u)'$ is given in this way by a unique m.

As it is shown in [7], every $m \in M_t(X,E')$ defines a β_o-continuous linear form on $C_b(X,E)$ by $u_m(f) = \int f dm$. Moreover the map $m \mapsto u_m$, from $M_t(X,E')$ to $(C_b(X,E), \beta_o)'$, is an algebraic isomorphism. Also it is shown in [11] that every $f \in C_b(X,E)$ is m-integrable, for every $m \in M_\tau(X,E'$, and the map u_m is β-continuous. Moreover, every element of $(C_b(X,E), \beta)'$ is given in this way for a unique $m \in M_\tau(X,E')$. For all unexplained terms on locally convex spaces we refer to [15] and [16].

Throughout the paper, unless it is stated explicitly othewise, X is a zero-dimensional Hausdorff topological space and E a Hausdorff locally convex space.

2 Extensions of Continuous Functions

The classical Tietze's extension Theorem states that, for a Hausdorff tropological space X, the following are equivalent: 1) X is normal.
2) For every closed subset Y of X and each continuous function $f : Y \to \mathbb{R}$, which is bounded (equivalently for which $f(X)$ is relatively compact), there exists a continuous extension $\bar{f} : X \to \mathbb{R}$ such that $\sup\{|f(x)| : x \in Y\} = \sup\{|\bar{f}(x)| : x \in X\}$
In this section we will examine the extension problem when we replace $\mathbb{R}$ by a complete non-Archimedean locally convex space E.

Lemma 2.1 *Let E be a Hausdorff locally convex space, $E \neq \{0\}$. If X is a Hausdorff topological space such that, for any closed subset Y of X and any $f \in C_{rc}(Y,E)$, there exists a continuous extension $\bar{f} : X \to E$ of f, then X is ultranormal.*

Proof: Let A, B be disjoint closed subsets of X and let a be a nonzero element of E. The function $f : A \cup B \to E, f(x) = 0$ if $x \in A$ and $f(x) = a$ if $x \in B$ is continuous. If g is a continuous extension of f and V a clopen neighborhood of zero in E not containing a, then $g^{-1}(V)$ is a a clopen subset of X containing A and disjoint from B, which proves that X is ultranormal.

Assume now that Y is a closed subset of X and that either Y is compact or X is ultranormal. In both cases, for every clopen in Y subset A of Y there exists a clopen subset B of X with $A = B \cap Y$. By [16], Corollary 5.23, there exists a family $(A_i)_{i\in I}$ of clopen in Y subsets of Y such that the family $\{\chi_{A_i} : i \in I\}$ of the corresponding characteritic functions is an orthonormal basis in $C_{rc}(Y)$ for the tropology of uniform convergence on $C_{rc}(Y)$. For each $i \in I$, choose a clopen subset $\tilde{A}_i$ of X whose intersection with Y is A_i. Then, as it is shown in the proof of Theorem 5.24 in [16], there exists a linear isometry $S : C_{rc}(Y) \to C_{rc}(X)$ such that $f(\chi_{A_i}) = \chi_{\tilde{A}_i}$ and Sg is an extension of g for every $g \in C_{rc}(Y)$.

Theorem 2.2 *Let $X, Y, (A_i)_{i\in I}$ and S be as above and assume that E is polar and complete. Then, there exists a linear map*

$$T : C_{rc}(Y, E) \to C_{rc}(X, E)$$

such that Tf is an extension of f and $\|Tf\|_p = \|f\|_p$ for all $f \in C_{rc}(Y, E)$ and all polar $p \in cs(E)$. Moreover, if $f = \sum_{i\in I} \chi_{A_i} s_i$, then $Tf = \sum_{i\in I} \chi_{\tilde{A}_i} s_i$ where convergence of the sums is with respect to the corresponding topologies of uniform convergence.

Proof: Claim I: If J is a finite subset of I, $f = \sum_{i\in J} \chi_{A_i} s_i$, $h = \sum_{i\in I} \chi_{\tilde{A}_i} s_i$, $s_i \in E$, then $\|h\|_p \le \|f\|_p$ (and hence $\|h\|_p = \|f\|_p$)
Indeed, given $\epsilon > 0$, there exists $x \in X$ with $p(h(x)) > \|h\|_p - \epsilon$. As p is polar, there exists $\phi \in E'$, $|\phi| \le p$, such that $|\phi(h(x))| > \|h\|_p - \epsilon$. Since $S\left(\sum_{i\in J} \phi(s_i)\chi_{A_i}\right) = \sum_{i\in J} \phi(s_i)\chi_{\tilde{A}_i}$, we have that

$$\|f\|_p \ge \|\sum_{i\in J} \phi(s_i)\chi_{A_i}\| = \|\sum_{i\in J} \phi(s_i)\chi_{\tilde{A}_i}\| \ge |\phi(h(x)| > \|h\|_p - \epsilon,$$

and the claim follows.
Claim II : If G is the subspace of all $f \in C_{rc}(Y, E)$ which can be written in the form $f = \sum_{i\in J} \chi_{A_i} s_i$, where all but a finite number of the s_i are zero, then G is τ_u-dense in $C_{rc}(Y, E)$.
To show this, we first observe that every $f \in G$ can be written uniquely in the form $f = \sum_{i\in J} \chi_{A_i} s_i$. In fact assume that $f = \sum_{i\in J_1} \chi_{A_i} s_i = \sum_{i\in J_2} \chi_{A_i} u_i$, where J_1, J_2 are finite subsets of I. We may assume that $J_1 = J_2 = J$. For each $\phi \in E'$, we have that $\sum_{i\in J} \phi(s_i)\chi_{A_i} = \sum_{i\in J} \phi(u_i)\chi_{\tilde{A}_i}$ and so $\phi(s_i) = \phi(u_i)$, for all $i \in J$, which implies that $s_i = u_i$ since E is Hausdorff and polar. Let now $f \in C_{rc}(Y, E)$ and a polar $p \in cs(E)$. There exist a finite clopen partition $\{D_1, \dots, D_n\}$ of Y and $x_i \in D_i$ such that $\|f - \sum_{k=1}^{n} \chi_{D_k} f(x_k)\|_p < 1$. Let A be a clopen subset of Y. Then $\chi_A = \sum_{i\in I} \alpha_i \chi_{A_i}$, $\alpha_i \in \mathbb{K}$, and so $\chi_A s = \sum_{i\in I} \alpha_i \chi_{A_i} s$ for all $s \in E$. To finish the proof of our claim, it suffices to prove that every $\chi_A s$ is in the closure of G in $C_{rc}(Y, E)$. So let q be a polar continuous seminorm on E and $\epsilon > 0$. There exists a finite subset J of I such that $\|\chi_A s - \sum_{i\in J} \alpha_i \chi_{A_i} s\|_q = q(s)\|\chi_A - \sum_{i\in J} \alpha_i \chi_{A_i}\|_q < \epsilon$, which proves that $\chi_A s \in \bar{G}$. This completes the proof of our claim.

Claim III: There exists a continuous linear map $T : C_{rc}(Y, E) \to C_{rc}(X, E)$ such that $T(f) = \sum_{i\in I} \chi_{\bar{A}_i} s_i$ for $f = \sum_{i\in I} \chi_{A_i} s_i$ in G. Indeed, define

$$T : G \to C_{rc}(X, E), \quad T(\sum_{i\in I} \chi_{A_i} s_i) = \sum_{i\in I} \chi_{\bar{A}_i} s_i.$$

Then T is well defined and linear. Moreover $\|Tf\|_p = \|f\|_p$ for each $f \in G$ and each polar $p \in cs(E)$. Since E is complete, the space $C_{rc}(X, E)$, with the topology of uniform convergence τ_u, is complete and hence (by Claim II) there exists a unique continuous extension of T to all of $C_{rc}(Y, E)$. We denote also by T this extension. If p is a polar continuous seminorm on E and $f \in C_{rc}(Y, E)$, then there exists a net (f_δ) in G converginmg to f. Thus $\|Tf\|_p = \lim \|Tf_\delta\|_p = \lim \|f_\delta\|_p = \|f\|_p$. Since Tf_δ is an extension of f_δ, it follows that Tf is an extension of f. This completes the proof.

For $p \in cs(E)$, let $M_{k,p}(X, E')$ be the space of all $m \in M_p(X, E')$ which have a compact support, i.e. there exists a compact subset Y of X such that $m(A) = 0$ if A is disjoint from Y. Let $m \in M_p(X, E')$, where $p \in cs(E)$. We will denote also by p the unique continuous extension of p to all of $\hat{E}$. If $\phi \in E'$ is such that $|\phi| \le p$, then there exists a unique continuous extension $\hat{\phi}$ of ϕ to all of $\hat{E}$. For each $A \in K(X)$, let $\hat{m}(A)$ be the continuous extension of $m(A)$. Then $\hat{m} \in M_p(X, \hat{E}')$ and $\hat{m}_p(A) = m_p(A)$. In fact, it is clear that $m_p(A) \le \hat{m}_p(A)$. On the other hand, let B be contained in A and let $s \in \hat{E}, s \neq 0$. If $\hat{m}(B)s \neq 0$, then there exists $u \in E$ with $p(s - u) < p(s)$ and $|\hat{m}(B)(s - u)| < |\hat{m}(B)s|$ Now $p(s) = p(u)$ and $|\hat{m}(B)s| = |\hat{m}(B)u|$. It follows easily from this that $\hat{m}_p(A) \le m_p(A)$, and the claim follows. It is also clear that $\hat{m} \in M_{t,p}(X, \hat{E}')$ if $m \in M_{t,p}(X, E')$.

As an application of the preceding Theorem , we get the following

Theorem 2.3 *Assume that E is polar and let p be a polar continuous seminorm on E. If we consider on $M_p(X, E')$ the norm $\|m\|_p = m_p(X)$, then $M_{t,p}(X, E')$ coincides with the closure of $M_{k,p}(X, E')$ in $M_p(X, E')$.*

Proof: Let $m \in M_p(X, E')$ be in the closure of $M_{k,p}(X, E')$. Given $\epsilon > 0$, choose $\bar{m} \in M_{k,p}(X, E')$ such that $\|m - \bar{m}\|_p < \epsilon$. Let Y be a compact support for $\bar{m}$. If $A \in K(X)$ is disjoint from Y, then for $B \subset A$ and $s \in E$ we have $|m(B)s| = |[m(B) - \bar{m}(B)]s| \le \|m - \bar{m}\|_p p(s)$ and so $m_p(A) \le \epsilon$, which proves that $m \in M_{t,p}(X, E')$. Conversely, let $m \in M_{t,p}(X, E')$. Then $\hat{m} \in M_{t,p}(X, \hat{E}')$. Let Y be a compact subset of X such that $m_p(A) = \hat{m}_p(A) \le \epsilon$ if A is disjoint from Y. Since $\hat{E}$ is complete and polar, there exists a linear map $S : C_{rc}(Y, \hat{E}) \to C_{rc}(X, \hat{E})$ such that, for each $f \in C_{rc}(Y, \hat{E}), Sf$ is an extension of f and $\|Sf\|_q = \|f\|_q$ for each continuous polar seminorm q on $\hat{E}$. Define

$$\phi : C_{rc}(X, \hat{E}) \to \mathbb{K}, \phi(f) = \int S(f|Y)d\hat{m}.$$

Then

$$|\phi(f)| \le m_p(X)\|S(f|Y)\|_p = m_p(X)\|f\|_{Y,p} \le m_p(X)\|f\|_p.$$

Hence, there exists $\mu \in M_p(X, \hat{E}')$ such that $\phi(f) = \int f d\mu$ for all $f \in C_{rc}(Y, \hat{E})$. Then Y is a support set for μ. Let $\bar{m} : K(X) \to E', \bar{m}(A) = \mu(A)|E$. Then $\bar{m} \in M_{k,p}(X, E')$. Finally, if $|\lambda| > 1$, then $\|\bar{m} - m\| \leq \epsilon|\lambda|$. Indeed, let $s \in E$ with $p(s) \leq 1$ and let $A \in K(X)$. If $h = S((\chi_A s)|Y)$ and $g = \chi_A s - h$, then $g = 0$ on Y and $\|g\|_p \leq 1$. Let $\mu \in \mathbb{K}, 0 < |\mu| < \epsilon/m_p(X)$. The set $V = \{x \in X : p(g(x)) > |\mu|\}$ is clopen and does not meet Y. Thus

$$\left|\int_V gdm\right| \leq m_p(V) \leq \epsilon, \quad \left|\int_{X\setminus V} gdm\right| \leq |\mu| m_p(X) \leq \epsilon.$$

Therefore $|m(A)s - \bar{m}(A)s| = |\int gdm| \leq \epsilon$. It follows that $\|m - \bar{m}\| \leq \epsilon|\lambda|$, which completes the proof.

3 The Completion of $(C_b(X, E), \beta_o)$

Let $C_{b,k}(X, E)$ be the space of all bounded E-valued functions on X whose restriction to every compact subset of X is continuous. For $p \in cs(E)$, let $\bar{\beta}_{o,p}$ be the locally convex topology on $C_{b,k}(X, E)$ generated by the seminorms $f \mapsto \|hf\|_p, h \in B_o(X)$. We define $\bar{\beta}_o$ to be the projective limit of the topologies $\bar{\beta}_{o,p}, p \in cs(E)$. For a sequence (K_n) of compact subsets of X and a sequence (d_n) of positive numbers, with $d_n \to \infty$, we denote by $W_{k,p}(K_n, d_n)$ the set $\bigcap_{n=1}^{\infty}\{f \in C_{b,k}(X, E) : \|f\|_{K_n,p} \leq d_n\}$. As in the case of β_o (see [7], p. 193), it can be shown that each $W_{k,p}(K_n, d_n)$ is a $\bar{\beta}_{o,p}$-neighborhood of zero. We also have the following Theorem whose proof is analogous to the proof of Proposition 2.6 in [7].

Theorem 3.1 *The sets of the form $W_{k,p}(K_n, |\lambda_n|)$, where (K_n) is an increasing sequence of compact subsets of X and (λ_n) a sequence in $\mathbb{K}$ with $0 < |\lambda_n| < |\lambda_{n+1}| \to \infty$, form a base at zero for $\bar{\beta}_{o,p}$.*

Theorem 3.2 *Let $p \in cs(E)$ and let W be an absolutely convex subset of $C_{b,k}(X, E)$. Then*
(1). If W is a $\bar{\beta}_{o,p}$-neighborhood of zero, then for every $r > 0$ there exist a compact subset Y of X and $\epsilon > 0$ such that

$$\{f \in C_{b,k}(X, E) : \|f\|_p \leq r, \|f\|_{Y,p} \leq \epsilon\} \subset W.$$

(2). If E is complete and polar and p a polar seminorm, then the converse holds in (1).

Proof: (1). It follows from the preceding Theorem.
(2). Assume that E is complete and polar, p is a polar seminorm and the condition holds in (1). Then, given $|\lambda| > 1$, there exist an increasing sequence (K_n) of compact subsets of X and a decreasing sequence (ϵ_n) of positive numbers such that $V_n \cap \lambda^n V \subset W$, where

$$V_n = \{f \in C_{b,k}(X, E) : \|f\|_{K_n,p} \leq \epsilon_n\}, V = \{f \in C_{b,k}(X, E) : \|f\|_p \leq 1\}.$$

Set $W_1 = V_1 \cap [\cap_{n=1}^{\infty}(V_{n+1} + \lambda^n V)]$. As in the proof of Theorem 2.8 in [7], we have that $W_1 \subset W$. Let now $\lambda_1 \in \mathbb{K}, 0 < |\lambda_1| < \min\{1, \epsilon_1\}$ and let $\lambda_n = \lambda^{n-1}$ for $n > 1$. We will finish the proof by showing that $W_2 = W_{k,p}(K_n, |\lambda_n|) \subset W_1$. So let $f \in W_2$. Then $f \in V_1$. Let m be a positive integer. There exists a linear map $T : C(K_{m+1}, E) \to C_{rc}(X, E)$ such that, for every $g \in C(K_{m+1}, E), Tg$ is an extension of g and $\|Tg\|_q = \|g\|_q$ for every polar $q \in cs(E)$. Let $g = T(f|K_{m+1}), h = f - g$. Then $h = 0$ on K_{m+1} and so $h \in V_{m+1}$. Also $\|g\|_p = \|f|K_{m+1}\|_p \leq |\lambda|^m$ and so $f \in V_{m+1} + \lambda^m V$, which proves that $f \in W_1$. This clearly completes the proof.

In the following Theorem, for each $p \in cs(E)$, we will denote also by p the unique continuous extension of p to all of $\hat{E}$.

Theorem 3.3 *If E is polar, then $(C_{b,k}(X, \hat{E}), \bar{\beta}_o)$ coincides with the completion of $(C_b(X, E), \beta_o)$.*

Proof: Claim I: $C_b(X, E)$ is β_o-dense in $C_b(X, \hat{E})$. Indeed, let W be a convex β_o-neighborhood of zero in $C_b(X, \hat{E})$. Since β_o is coarser than τ_u, there exists $p \in cs(E)$ such that $W_1 = \{f \in C_b(X, \hat{E}) : \|f\|_p \leq 1\} \subset W$. Let $A \in K(X)$ and $s \in \hat{E}$. Choose $w \in E$ with $p(s - w) < 1$. Then $\chi_A s - \chi_A w \in W_1$, which proves that $\chi_A s$ belongs to the closure of $C_b(X, E)$ in $C_b(X, \hat{E})$. Since the space spanned by the functions $\chi_A s, A \in K(X), s \in \hat{E}$, is β_o-dense in $C_b(X, \hat{E})$, our claim follows.
Let now W be a convex $\bar{\beta}_o$-neighborhood of zero in $C_{b,k}(X, \hat{E})$ and let $f \in C_{b,k}(X, \hat{E})$. There exists a polar continuous seminorm p on E such that W is a $\bar{\beta}_{o,p}$-neighborhood. In view of the preceding Theorem, there exist a compact subset Y of X and $\epsilon > 0$ such that

$$\{g \in C_{b,k}(X, \hat{E}) : \|g\|_p \leq \|f\|_p, \|g\|_{Y,p} \leq \epsilon\} \subset W.$$

Let $h \in C_b(X, \hat{E})$ be an extension of $f|Y$ such that $\|h\|_p = \|f\|_{Y,p}$. Now $\|f - h\|_p \leq \|f\|_p$ and $f = h$ on Y, which implies that $f - h$ is in W. Thus $C_b(X, \hat{E})$ is $\bar{\beta}_o$-dense in $C_{b,k}(X, \hat{E})$, which, combined with Claim I, implies that $C_b(X, E)$ is $\bar{\beta}_o$-dense in $C_{b,k}(X, \hat{E})$.
Claim II: $(C_{b,k}(X, \hat{E}), \bar{\beta}_o)$ is complete. In fact, let (f_δ) be a $\bar{\beta}_o$-Cauchy net. For each $x \in X, (f_\delta(x))$ is a Cauchy net in $\hat{E}$. Thus we get a function $f : X \to \hat{E}, f(x) = \lim f_\delta(x)$. Since $f_\delta \to f$ uniformly on compact subsets of X, it follows that $f|Y$ is continuous for every compact set Y. Also, f is bounded. Indeed, suppose that there exist $p \in cs(E)$ and a sequence (x_n) of elements of X such that $p(f(x_n)) < p(f(x_{n+1})) \to \infty$. The set $W = \{g \in C_{b,k}(X, \hat{E}) : p(g(x_n)) \leq p(f(x_n))/2\}$ is a $\bar{\beta}_{o,p}$-neighborhood of zero. Thus, there exists δ_o such that $f_\delta - f_{\delta_o} \in W$ for $\delta \geq \delta_o$. It follows from this that $p(f(x_n) - f_{\delta_o}(x_n)) \leq p(f(x_n))/2$. Thus $p(f_{\delta_o}(x_n)) = p(f(x_n)) \to \infty$, a contradiction. By the above $f \in C_{b,k}(X, \hat{E})$. Moreover $f_\delta \to f$ in $C_{b,k}(X, \hat{E})$, which completes the proof.

Corollary 3.4 *If E is polar, then $(C_b(X, E), \beta_o)$ is complete iff E is complete and every bounded E-valued f on X such that $f|Y$ is continuous, for every compact subset Y of X, is continuous on X.*

Theorem 3.5 *If E is polar and complete, then $(C_b(X,E),\beta_o)$ is complete iff it is quasicomplete.*

Proof: Assume that $(C_b(X,E),\beta_o)$ is quasicomplete and let $f \in C_{b,k}(X,E)$. For each compact subset K of X there exists f_K in $C_b(X,E)$ such that $f_K = f$ on K and $\|f\|_{K,p} = \|f_K\|_p$ for each continuous polar seminorm p on E. The set $\{f_K : K \subset X, K\text{compact}\}$ is contained in the uniformly bounded subset D of $C_b(X,E)$ consisting of all g with $\|g\|_p \leq \|f\|_p$ for all $p \in cs(E)$, p polar. On D, β_o coincides with the topology τ_k of compact convergence. Ordering the family $\mathcal{K}$ of all compact subsets of X by set inclusion, we get a net $(f_K)_{K\in\mathcal{K}}$ in $C_b(X,E)$ which is τ_k-Cauchy and hence β_o-Cauchy. Since D is β_o-bounded, there exists $g \in C_b(X,E)$ such that the net (f_K) is β_o-cnvergent to g. But then $g(x) = \lim f_K(x) = f(x)$ for all x and so $f = g \in C_b(X,E)$. Now the result follows from the preceding Corollary.

Recall that a topological space Y is called a P-space if every zero set is open. In case Y is zero-dimensional, Y is a P-space iff every $\mathbb{K}$-zero set is open, equivalently iff every countable intersection of clopen sets is clopen.

Theorem 3.6 *If X is a P-space, then $(C_b(X,E),\beta_o)$ is sequentially-complete iff E is sequentially-complete.*

Proof: Assume that $(C_b(X,E),\beta_o)$ is sequentially-complete and let (s_n) be a Cauchy sequence in E. The sequence $(g_n), g_n(x) = s_n$ for all $x \in X$, is β_o-Cauchy. If (g_n) is β_o-cnvergent to g, then $g(x) = \lim s_n$ and so E is sequentially-complete. Conversely, let E be sequentially-complete and let (f_n) be a β_o-Cauchy sequence in $C_b(X,E)$. Since β_o is finer than the topology of simple convergence, the limit $f(x) = \lim f_n(x)$ exists in E for each $x \in X$. Then f is bounded. Indeed, assume that there exists a $p \in cs(E)$ such that $\|f\|_p = \infty$. Choose a sequence (a_n) of elements of X such that $p(f(a_n)) > n$ for all n. The set

$$W = \{g \in C_b(X,E) : p(g(a_n)) \leq n, n \in \mathbb{N}\}$$

is a β_o-neighborhood of zero. Let n_o be such that $f_n - f_{n_o} \in W$ for $n \geq n_o$. For $n \geq n_o$ we have that $p(f_n(a_k) - f_{n_o}(a_k)) \leq k$ and so $p(f(a_k) - f_{n_o}(a_k)) \leq k$, which implies that $p(f_{n_o}(a_k)) = p(f(a_k)) > k$, for all k, a contradiction since f_{n_o} is bounded. Also f is continuous. In fact, let $x \in X$ and let D be a clopen neighborhood of $f(x)$ in E. Each $f_n^{-1}(D)$ is a clopen neighborhood of x and so $V = \cap V_n$ is a neighborhood of x since X is a P-space. Moreover, for $y \in V$, $f(y) \in \bar{D} = D$, which proves the continuity of f at x. Moreover, since β_o has a base at zero consisting of sets which are closed with respect to the topology of simple convergence, it follows that (f_n) is β_o-convegent to f, and this completes the proof.

4 Product Measures

Let $B_{ou}(X)$ be the family of all $\phi \in B_o(X)$ for which $|\phi|$ is upper semicontinuous. As it is shown in [12], if $|\lambda| > 1$, then for every $\phi \in B_o(X)$ there exists $\psi \in B_{ou}(X)$ such that $|\psi| \leq |\phi| \leq \lambda\psi|$. Thus β_o is defined by the seminorms $f \mapsto \|\phi f\|, \phi \in B_{ou}(X), p \in cs(E)$. If Y is another Hausdorff zero-dimensional topological space, then for each $\phi_1 \in B_{ou}(X)$ and each $\phi_2 \in B_{ou}(Y)$, the function $\phi_1 \times \phi_2$, which is defined on $X \times Y$ by $\phi_1 \times \phi_2(x,y) = \phi_1(x)\phi_2(y)$, is in $B_{ou}(X \times Y)$. Also, given $\phi \in B_{ou}(X \times Y)$, there exist $\phi_1 \in B_{ou}(X), \phi_2 \in B_{ou}(Y)$ such that $|\phi_1 \times \phi_2| \geq |\phi|$. Thus the topology β_o on $C_b(X,E)$ is defined by the seminorms $f \mapsto \sup_{x\in X, y\in Y} p(\phi_1(x)\phi_2(y)f(x,y))$, where $\phi_1 \in B_{ou}(X), \phi_2 \in B_{ou}(Y), p \in cs(E)$.

Theorem 4.1 *Let X, Y be zero-dimensional Hausdorff topological spaces. If G is the subspace of $C_b(X \times Y, E)$ spanned by the functions $\chi_{A\times B}s, A \in K(X), B \in K(Y), s \in E$, then G is β_o-dense in $C_b(X \times Y, E)$.*

Proof: Let $p \in cs(E), \phi_1 \in B_{ou}(X), \phi_2 \in B_{ou}(Y), W = \{f \in C_b(X \times Y, E) : p(\phi_1(x)\phi_2(y)f(x,y)) \leq 1\}$. Let $f \in C_b(X \times Y, E)$. The set

$$D = \{(x,y) : p(\phi_1(x)\phi_2(y)f(x,y)) \geq 1/2\}$$

is compact. If D_1, D_2 are the projections of D on X, Y, respectively, then $D \subset D_1 \times D_2$. Choose $d > \|\phi_1\|, \|\phi_2\|$ and let $x \in D_1$. There exists $y \in Y$ such that $(x,y) \in D$ and hence $\phi_1(x) \neq 0$. The set $Z_x = \{z \in X : |\phi_1(z)| < 2|\phi_1(x)|\}$ is open and contains x. Using the compactness of D_2, we find a clopen neighborhood A_x of x contained in Z_x such that $p(f(z,y) - f(x,y)) < 1/d^2$ for all $z \in A_x, y \in D_2$. Because of the compactness of D_1, there are $x_1, \dots, x_m$ in D_1 such that $D_1 \subset \bigcup_{k=1}^m A_{x_k}$. Let $A_1 = A_{x_1}, A_{k+1} = A_{x_{k+1}} \setminus \bigcup_{i=1}^k A_{x_i}, k = 1, \dots, m-1$. Keeping those of the A_k which are not empty, we may assume that each $A_k \neq \emptyset$. For each $1 \leq k \leq m$, there are pairwise disjoint clopen sets $B_{k1}, \dots, B_{kn_k}$ of Y, covering D_2, and $y_{kj} \in B_{kj}$ such that $p(f(x_k,y) - f(x_k,y_{kj})) < 1/(2d^2)$ if $y \in B_{kj}$. Let now $h = \sum_{k=1}^m \sum_{j=1}^{n_k} \chi_{A_k\times B_{kj}} f(x_k, y_{kj})$. Then $h \in G$. Moreover, $p(\phi_1(x)\phi_2(y)(f(x,y) - h(x,y)) \leq 1$ for all (x,y). To prove this, we consider the three possible cases:
Case I. $x \notin \bigcup_{k=1}^m A_k$. Then $h(x,y) = 0$. Also $(x,y) \notin D$ and so $p(\phi_1(x)\phi_2(y)f(x,y)) \leq 1/2$.
Case II. $x \in A_k, y \in D_2$. There exists j with $y \in B_{kj}$. Now $p(f(x,y) - f(x_k,y)) < 1/d^2$ and $p(f(x_k,y) - f(x_k,y_{kj})) < \frac{1}{2d^2}$, which implies that $p(\phi_1(x)\phi_2(y)(f(x,y) - h(x,y)) \leq 1$.
Case III. $x \in A_k, y \notin D_2$. Then $(x,y) \notin D$ and so $p(\phi_1(x)\phi_2(y)f(x,y)) \leq 1/2$. If $h(x,y) \neq 0$, then $y \in B_{kj}$ for some j, and so $h(x,y) = f(x_k,y_{kj}), p(f(x_k,y) - f(x_k,y_{kj})) \leq \frac{1}{2d^2}$. Since $x \in A_{x_k}$, we have that $|\phi_1(x)| < 2|\phi_1(x_k)|$ and thus $p(\phi_1(x)\phi_2(y)f(x_k,y)) \leq 2p(\phi_1(x_k)\phi_2(y)f(x_k,y)) \leq 1$ since $(x_k,y) \notin D$. It follows that $p(\phi_1(x)\phi_2(y)h(x,y)) \leq 1$ and our claim follows. This clearly completes the proof.

Theorem 4.2 *If $\mu \in M_\tau(X)$ and $m \in M_{t,p}(Y,E')$, then there exists a unique $\bar{m} \in M_t(X \times Y, E')$ such that $\bar{m}(A \times B) = \mu(A)m(B)$ for each $A \in K(X)$ and each $B \in K(Y)$. Moreover, $\bar{m} \in M_{t,p}(X \times Y, E')$.*

Proof: By [12], Theorem 4.6, there exists a linear map

$$\omega : M = (C_b(X), \beta_o) \otimes (C_b(Y,E), \beta_o) \to (C_b(X \times Y, E), \beta_o)$$

such that $\omega(g \otimes f) = g \times f$, for all $g \in C_b(X), f \in C_b(Y,E)$, where $(g \times f)(x,y) = g(x)f(y)$, and $\omega : M \to \omega(M)$ is a topological isomorphism. In view of the preceding Theorem, $\omega(M)$ is β_o-dernse in $C_b(X \times Y, E)$. The bilinear map

$$T : (C_b(X), \beta_o) \times (C_b(Y,E), \beta_o) \to \mathbb{K}, \quad T(g,f) = \left(\int g d\mu\right)\left(\int f dm\right)$$

is continuous. Hence we have a continuous linear map $\phi : M \to \mathbb{K}, \phi(g \otimes f) = T(g,f)$. Since $\omega : M \to \omega(M)$ is a topological isomorphism, it follows that the linear map $\psi : \omega(M) \to \mathbb{K}, \psi = \phi \circ \omega^{-1}$, is β_o-continuous on $\omega(M)$. As $\omega(M)$ is β_o-dense in $C_b(X \times Y, E)$, there is a continuous extension $\tilde{\psi}$ of ψ to all of $C_b(X \times Y, E)$. Thus, there exists $\bar{m} \in M_t(X \times Y, E)$ such that $\tilde{\psi}(h) = \int h d\bar{m}$ for all $h \in C_b(X \times Y, E)$. In particular, for $g \in C_b(X), f \in C_b(Y,E)$, we have $\psi(g \times f) = \int (g \times f) d\bar{m} = (\int g d\mu)(\int f dm)$. If $A \in K(X), B \in K(Y), s \in E$ and $h = \chi_{A \times B} s = \chi_A \times (\chi_B s)$, then

$$\bar{m}(A \times B)s = \tilde{\psi}(h) = \mu(A)m(B)s$$

and so $\bar{m}(A \times B)) = \mu(A)m(B)$.
Let now $m_1 \in M_t(X \times Y, E')$ be such that $m_1(A \times B)) = \mu(A)m(B)$ for all $A \in K(X), B \in K(Y)$. Consider the β_o-continuous linear forms $\phi_1(h) = \int h d\bar{m}, \phi_2(h) = \int h dm_1$. If G is as in the proof of the preceding Theorem, then $\phi_1 = \phi_2$ on G and hence $\phi_1 = \phi_2$ since G is β_o-dense in $C_b(X \times Y, E)$. Thus $\bar{m} = m_1$. Finally, assume that $m \in M_{t,p}(X, E')$. There are $\phi_1 \in B_{ou}(X)$ and $\phi_2 \in B_{ou}(Y)$ such that $|\int g dm| \le \|\phi_1 g\|$ and $|\int f dm| \le \|\phi_2 f\|_p$ for all $g \in C_b(X), f \in C_b(Y,E)$. Thus, for $h = g \times f$, we have that $|\int h d\bar{m}| \le \|(\phi_1 \times \phi_2)h\|_p$. Since the map $h \mapsto \|(\phi_1 \times \phi_2)h\|_p$ is a β_o-continuous seminorm on $C_b(X \times Y, E)$, it follows that $|\int h d\bar{m}| \le \|(\phi_1 \times \phi_2)h\|_p$ for all $h \in C_b(X \times Y, E)$. In particular, for $D \in K(X \times Y)$ and $s \in E$, we have

$$|\bar{m}(D)s| \le p(s) \sup_{(x,y) \in X \times Y} |\phi_1(x)\phi_2(y)| \le p(s)\|\phi_1 \times \phi_2\|.$$

Thus, $\bar{m}_p(X \times Y) \le \|\phi_1 \times \phi_2\| = \|\phi_1\|\phi_2\|$. This completes the proof.

Definition 4.3 *For $\mu \in M_\tau(X)$ and $m \in M_t(Y, E')$, we define by $\mu \times m$ the unique element $\bar{m}$ of $M_t(X \times Y, E')$ for which $\bar{m}(A \times B) = \mu(A)m)B)$ for $A \in K(X), B \in K(Y)$. We call this $\bar{m}$ the product of μ and m.*

Theorem 4.4 *Let $h \in C_b(X \times Y, E)$ and $m \in M_{t,p}(Y, E')$. Then the function*

$$g : X \to \mathbb{K}, \quad g(x) = \int_Y f(x,y) dm(y)$$

is bounded and continuous.

Proof: Without loss of generality, we may assume that $\|m\|_p \le 1$ and $\|f\|_p \le 1$. Let $\epsilon > 0$ and let D be a compact subset of Y such that $m_p(A) < \epsilon$ if A is disjoint from D. Let $x_o \in X$. For each $y \in D$ there are clopen neighborhoods V_y and W_y of y and x_o, respectively, such that $p(f(x,z) - f(x_o,y)) < \epsilon$ if $x \in W_y, z \in V_y$. Let $y_1, \dots, y_n$ in D be such that $D \subset V = \bigcup_{k=1}^n V_{y_k}$ and let $W = \bigcap_{k=1}^n W_{y_k}$. Then, for

$x \in W, y \in V$ we have that $p(f(x,y) - f(x_o,y)) \leq \epsilon$. It follows that , for $x \in W$, we have

$$|\int_V f(x,y)dm(y) - \int_V f(x_o,y)dm(y)| \leq \epsilon.$$

Also,

$$|\int_{Y\setminus V} f(x,y)dm(y)| \leq \|f\|_p m_p(Y \setminus V) \leq \epsilon \; \textit{and} \; |\int_{Y\setminus V} f(x_o,y)dm(y)| \leq \epsilon.$$

Thus, for $x \in W$, we have $|g(x) - g(x_o)| \leq \epsilon$, which proves thast g is continuous. Moreover $\|g\| \leq 1$.

Theorem 4.5 *Let $\mu \in M_\tau(X), m \in M_{t,p}(Y,E'), \bar{m} = \mu \times m$. If $h \in C_b(X \times Y, E)$, then $\int h d\bar{m} = \int_X [\int_Y h(x,y)dm(y)]d\mu(x)$.*

Proof: Define

$$\psi : C_b(X \times Y, E) \to \mathbb{K}, \quad \psi(f) = \int_X \int_Y f(x,y)dm(y)d\mu(x).$$

There are $\phi_1 \in B_{ou}(X), \phi_2 \in B_{ou}(Y)$ such that for every $g \in C_b(X)$ and every $f \in C_b(Y,E)$, we have

$$|\int g d\mu| \leq \|\phi_1 g\| \quad \textit{and} \quad |\int f dm| \leq \|\phi_2 f\|_p.$$

Now, for all $x \in X$, we have $|\int_Y h(x,y)dm(y)| \leq \sup_{y \in Y} |\phi_2(y)| p(h(x,y))$ and

$$|\int_X [\int_Y h(x,y)dm(y)]d\mu(x)| \leq \sup_{x\in X}[\sup_{y \in Y} |\phi_2(y)| p(h(x,y))]|\phi_1(x)| = \sup_{(x,y)\in X\times Y} |\phi_1 \times \phi_2(x,y)| p(h(x,y)).$$

Since $\phi_1 \times \phi_2 \in B_{ou}(X \times Y)$, it follows that ψ is β_o-continuous on $C_b(X \times Y, E)$. For $A \in K(X), B \in K(Y), f = \chi_{A \times B)} s = \chi_A \times (\chi_B s)$, we have

$$\psi(f) = \int_X [\int_Y \chi_A(x)\chi_B(y)dm(y)]d\mu(x) = \mu(A)m(B)s$$

and $\int f d\bar{m} = \mu(A)m(B)s$. Thus $\psi(f) = \int f d\bar{m}$ for $f \in G$, where G is as in Theorem 4.1. Since G is β_o-dense in $C_b(X \times Y, E)$, we have that $\psi(f) = \int f d\bar{m}$ for all $f \in C_b(X \times Y, E)$. This completes the proof.

5 (VR)-Integrals

Van Rooij defined in [16] integration of functions in $\mathbb{K}^X$ with respect to members μ of $M_\tau(X)$. His definition however cannot be applied for arbitrary μ in $M(X)$. Let $\mu \in M_\tau(X)$. He defined $N_\mu : X \to \mathbb{R}$ by $N_\mu(x) = \inf\{|\mu|(A) : x \in A \in K(X)\}$. Then N_μ is upper semicontinuous and, for every $\epsilon > 0$, the set $\{x \in X : N_\mu(x) \geq \epsilon\}$ is compact. For $A \in K(X)$ we have that $|\mu|(A) = \sup_{x\in A} N_\mu(x)$. For $f \in \mathbb{K}^X$, he defined $\|f\|_{N_\mu} = \sup_x |f(x)| N_\mu(x)$. If g is a $K(X)$-simple function, i.e. $g =$

$\sum_{k=1}^{n} \alpha_k \chi_{A_k}$, with $A_k \in K(X), \alpha_k \in \mathbb{K}$, he defined $\int g d\mu = \sum_{k=1}^{n} \alpha_k m(A_k)$. Van Rooij called an $f \in \mathbb{K}^X$ μ-integrable if there exists a sequence (g_n) of simple functions such that $\|f - g_n\|_{N_\mu} \to 0$. In this case, he called integral of f the $\lim \int g_n d\mu$. We will denote by $(VR) \int f d\mu$ the integral of f in his sense. It was proved in [10] that, for $\mu \in M_\tau(X)$, if f is μ-integrable in our sense, then f is also integrable in Van Rooij's sense and the two integrals coincide.

In this section we will assume that E is a normed space and we will define the integral $(VR) \int f dm$ of an f in E^X with respect to an $m \in M_t(X, E') = M_\tau(X, E')$. Most of the aguments we will use will be analogous to the ones used in [16] where scalar-valued measurers and functions in $\mathbb{K}^X$ are treated. Let $m \in M_t(X, E')$. As in [16], we define

$$N_m : X \to \mathbb{R}, N_m(x) = \inf\{|m|(A) : x \in A \in K(X)\},$$

where $|m| = m_{\|.\|}$. Then N_m is upper-semicontinuous and $|m|(A) = \sup_{x \in A} N_m(x)$ for each $A \in K(X)$.

Let $S(X, E)$ be the space of all E-valued $K(X)$-simple functions on X. For $h \in E^X$, we define $\|h\|_{N_m} = \sup_{x \in X} N_m(x)\|h(x)\|$.

Lemma 5.1 *If $m \in M_t(X, E')$ and $g = \sum_{k=1}^{n} \chi_{A_k} s_k \in S(X, E)$, then*

$$|\sum_{k=1}^{n} m(A_k) s_k| \leq \|g\|_{N_m} \leq \|g\| \|m\|.$$

Proof: Without loss of generality we may assume that the sets $A_1, \dots, A_n$ are pairwise disjoint. Since, for $A \in K(X)$ and $s \in E$, we have $|m(A)s| \leq \|s\| |m|(A) = \|s\| \sup_{x \in A} N_m(x)$, the Lemma follows.

We have the following easily established

Lemma 5.2 *Let $m \in M_t(X, E')$ and $f \in E^X$. Assume that there exists a sequence $(g_n) \subset S(X, E)$ such that $\|f - g_n\|_{N_m} \to 0$. Then: (1) The $\lim_{n\to\infty} \int g_n dm$ exists.*
(2) If (h_n) is another sequence in $S(X, E)$ such that $\|f - h_n\|_{N_m} \to 0$, then $\lim_{n\to\infty} \int g_n dm = \lim_{n\to\infty} \int h_n dm$.
(3) $|\lim_{n\to\infty} \int g_n dm| \leq \|f\|_{N_m} < \infty$.

Definition 5.3 *Let $m \in M_t(X, E')$. A function $f \in E^X$ is called (VR)-integrable with respect to m if there exists a sequence $(g_n) \subset S(X, E)$ such that $\|f - g_n\|_{N_m} \to 0$. In this case we define*

$$(VR) \int f dm = \lim_{n\to\infty} \int g_n dm.$$

Let now $m \in M_t(X, E')$ and let

$$\mathcal{S}_m = \{A \subset X : \chi_A s \text{ is (VR)-integrable for all } s \in E\}.$$

As in [16], Lemma 7.3, we have the following

Lemma 5.4 *Let $m \in M_t(X, E')$ and $A \subset X$. Then $A \in \mathcal{S}_m$ iff, for every $\epsilon > 0$, there exists $B \in K(X)$ such that $N_m < \epsilon$ on $A \Delta B = (A \setminus B) \cup (B \setminus A)$.*

Proof: Assume that $A \in \mathcal{S}_m$ and let s be a non-zero element of E. Let $g \in S(X, E)$ be such that $\|\chi_A s - g\|_{N_m} < \epsilon\|s\|$. If $B = \{x : \|g(x) - s\| < \|s\|\}$, then $B \in K(X)$ and $\|g(x) - \chi_B(x)s\| \leq \min\{\|g(x)\|, \|g(x) - s\|\} \leq \|g(x) - \chi_A(x)s\|$ and so

$$\|\chi_A s - \chi_B s\|_{N_m} \leq \max\{\|\chi_A s - g\|_{N_m}, \|\chi_B s - g\|_{N_m}\} = \|\chi_A s - g\|_{N_m} < \epsilon\|s|,$$

which implies that $N_m < \epsilon$ on $A\Delta B$.
Conversely, suppose that the condition is satisfied and let s be a non-zero element of E. Choose $B \in K(X)$ such that $N_m < \epsilon/\|s\|$ on $A\Delta B$. Then $\|\chi_A s - \chi_B s\|_{N_m} \leq \epsilon$ which completes the proof.

We can easily prove the following

Lemma 5.5 *Let $m \in M_t(X, E')$. Then: (1) For each $A \in \mathcal{S}_m$, the complement A^c is also in $\mathcal{S}_m$.*
(2) If $A_1, A_2 \in \mathcal{S}_m$, then $A_1 \cup A_2$ and $A_1 \cap A_2$ are in $\mathcal{S}_m$.
(3) $K(X) \subset \mathcal{S}_m$.
(4) $A \in \mathcal{S}_m$ iff, for each $\epsilon > 0$, there exists $B \in K(X)$ such that $A \cap X_{m,\epsilon} = B \cap X_{m,\epsilon}$, where $X_{m,\epsilon} = \{x : N_m(x) \geq \epsilon\}$.

For $m \in M_t(X, E')$, we denote by τ_m the zero-dimensional topology on X having $\mathcal{S}_m$ as a base. Clearly τ_m is finer than the topology τ of X. We denote by X_m the set X equipped with the topology τ_m.

Theorem 5.6 *Let $m \in M_t(X, E')$. Then $X_{m,\epsilon}$ is τ_m-compact for each $\epsilon > 0$.*

Proof: It suffices to show that every cover $\mathcal{U}$ of $X_{m,\epsilon}$ by sets in $\mathcal{S}_m$ has a finite subcover. Without loss of generality, we may assume that $A_1 \cup A_2$ is in $\mathcal{U}$ if $A_1, A_2 \in \mathcal{U}$. Since N_m is τ_m-upper semicontinuous, $X_{m,\epsilon}$ is τ_m-closed. Hence the family

$$\mathcal{V} = \{(V \cup Z)^c : V \in \mathcal{U}, Z \subset X^c_{m,\epsilon}, Z \in \mathcal{S}_m\}$$

is downwards directed to the empty set. Since $|m|$ is τ-additive, there exist $V \in \mathcal{U}, Z \subset X^c_{m,\epsilon}$ such that $|m|((V \cup Z)^c) < \epsilon$ and so $X_{m,\epsilon} \subset V \cup Z$, which implies that $X_{m,\epsilon} \subset V$, and we are done.

Since $X_{m,\epsilon}$ is τ_m-compact and τ is Hausdorff, it follows that $\tau = \tau_m$ on $X_{m,\epsilon}$.

Lemma 5.7 *For $m \in M_t(X, E')$, an $A \subset X$ is τ_m-clopen iff it is in $\mathcal{S}_m$.*

Proof: Asuume that A is τ_m-clopen. Then, for $\epsilon > 0$, the set $A \cap X_{m,\epsilon}$ is clopen in in $X_{m,\epsilon}$ for the topology induced by τ_m and hence for the topology induced by τ. Since $X_{m,\epsilon}$ is τ-compact, there exists $B \in K(X)$ such that $A \cap X_{m,\epsilon} = B \cap X_{m,\epsilon}$. The result now follows from Lemma 5.5.

Proposition 5.8 *If $m \in M_t(X, E')$ and $f \in E^X$, then f is τ_m-continuous iff $f|X_{m,\epsilon}$ is τ-continuous for each $\epsilon > 0$.*

Proof: Since $\tau = \tau_m$ on $X_{m,\epsilon}$, the necessity is clear. Conversely, assume that the condition is satisfied. If D is a clopen subset of E, then $f^{-1}(D) \cap X_{m,\epsilon}$ is clopen in $X_{m,\epsilon}$ for the topology induced on $X_{m,\epsilon}$ by τ. Since $X_{m,\epsilon}$ is τ-compact, there exists $A \in K(X)$ such that $A \cap X_{m,\epsilon} = f^{-1}(D) \cap X_{m,\epsilon}$. Thus $f^{-1}(D$ is τ_m-clopen by Lemma 5.5 and the result follows.

Theorem 5.9 *Let $m \in M_t(X, E')$. For a τ_m-clopen subset A of X, we define $\bar{m}(A)$ on E by $\bar{m}(A)s = (VR)\int \chi_A s dm$. Then : 1) $\bar{m}(A) \in E'$.*
2) $\bar{m}(A) \in M_t(X_m, E')$, $\|m| = \|\bar{m}\|$ and $|\bar{m}|(A) = |m|(A)$ for $A \in K(X)$.

Proof: 1) It follows from the inequality

$$|(VR)\int \chi_A s dm| \leq \sup_{x \in A} \|s\| N_m(x) \leq \|m\|\|s\|.$$

2) Clearly $\bar{m}$ is finitely additive. Let $\mathcal{A}$ be a family of τ_m-clopen sets which is downwards directed to the empty set and let $Y = X_{m,\epsilon}$. For each $A \in \mathcal{A}$, there exists $B \in K(X)$ such that $A \cap Y = B \cap Y$. Let

$$\mathcal{B} = \{B \in K(X) : \exists A \in \mathcal{A}, A \cap Y = B \cap Y\}.$$

Let $B_1, B_2 \in \mathcal{B}$ and let $A_1, A_2 \in \mathcal{A}$ such that $A_i \cap Y = B_i \cap Y$, for $i = 1, 2$. Let $A \in \mathcal{A}, A \subset A_1 \cap A_2$ and choose $B \in K(X)$ with $A \cap Y = B \cap Y$. If $D = A \cap B_1 \cap B_2$, then $A \cap Y = D \cap Y$ and so $D \in \mathcal{B}$, which proves that $\mathcal{B}$ is downwards directed. Moreover $\bigcap \mathcal{B} = \emptyset$. Indeed assume that $x \in \bigcap \mathcal{B}$. If $x \notin Y$, then there exists $Z \in K(X)$ containing x with $|m|(Z) < \epsilon$ and so Z is disjoint from Y. If $B \in \mathcal{B}$, then there exists $A \in \mathcal{A}$ with $A \cap Y = B \cap Y = (B \backslash Z) \cap Y$ and so $B \backslash Z \in \mathcal{B}$, a contradiction since $x \notin B \setminus Z$. Thus x must be in Y and so $x \in \cap \mathcal{B} = \bigcap_{B \in \mathcal{B}} B \cap Y$. Given $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ with $A \cap Y = B \cap Y$ and so $x \in A$, i.e. $x \in \bigcap \mathcal{A}$, a contradiction. Thus $\mathcal{B}$ is downwards directed to the empty set. Since $m \in M_t(X, E')$, there exists $B \in \mathcal{B}$ with $|m|(B) < \epsilon$. Let $A \in \mathcal{A}$ with $A \cap Y = B \cap Y = \emptyset$. If $x \in A$, then $x \notin Y$ and so $N_m(X) < \epsilon$. If G is a τ_m-clopen set contained in A, then for each $s \in E$ we have

$$|\bar{m}|(G)s| \leq \sup_{x \in G} \|s\| N_m(x) \leq \epsilon \|s|$$

and so $|\bar{m}|(A) \leq \epsilon$. This proves that $\bar{m} \in M_\tau(X_m, E') = M_t(X_m, E')$. Finally, let $A \in K(X)$. Clearly $|m|(A) \leq |\bar{m}|(A)$. On the other hand, let D be a τ_m-clopen subset of A. For each $s \in E$, we have

$$|\bar{m}(D)s| = |(VR)\int \chi_D s dm| \leq \sup_{x \in D} \|s\| N_m(x) \leq \|s\| |m|(A),$$

which proves that $|m|(A) \geq |\bar{m}|(A)$, and the result follows.

Proposition 5.10 *If $m \in M_t(X, E')$, then $N_{\bar{m}} = N_m$.*

Proof: Since $|m|(A) = |\bar{m}|(A)$ for $A \in K(X)$, it follows that $N_{\bar{m}} \leq N_m$. Assume that, for some $x \in X$, we have $N_{\bar{m}}(x) < \epsilon < N_m(x)$. There exists a τ_m-clopen set A containing x with $|\bar{m}|(A) < \epsilon$. Let $B \in K(X)$ such that $A \cap Y = B \cap Y, Y = \{y :$

$N_m(y) \geq \epsilon\}$. Then $x \in B$ and so $|m|(B) \geq N_m(x) > \epsilon$. Let $D \in K(X)$ contained in B and $s \in E$ be such that $|m(D)s|/\|s\| > \epsilon$. Then $|\bar{m}(D \cap A)s|/\|s\| \leq |\bar{m}|(D \cap A) < \epsilon$. Since $m(D) = \bar{m}(D)$, we have that $|m(D)s| = |m(D)s - \bar{m}(D \cap A)s| = |\bar{m}(D \setminus A)s| \leq \|s\| \sup_{y \in D \setminus A} N_m(y)$. But, if $y \in D \setminus A$, then $N_m(y) < \epsilon$, since $D \subset B$ and $A \cap Y = B \cap Y$, and so $|m(D)s| \leq \epsilon \|s\|$, a contradiction. This completes the proof.

Lemma 5.11 *Let $m \in M_t(X, E')$ and $g \in S(X_m, E)$. Then, for each $\epsilon > 0$, there exists $h \in S(X, E)$ such that $\|g - h\|_{N_m} \leq \epsilon$.*

Proof: If $g \neq 0$, there are paiwise disjoint τ_m-clopen sets $A_1, \ldots, A_n$ and non-zero elements $s_1, \ldots, s_n$ in E such that $g = \sum_{k=1}^n \chi_{A_k} s_k$. Let $\alpha = \min\{\|s_i\| : i = 1, \ldots, n\}$. For each i, choose $B_i \in K(X)$ with $N_m < \epsilon/\alpha$ on $A_i \Delta B_i$. Let $Z_1 = B_1, Z_{k+1} = B_{k+1} \setminus \bigcup_{i=1}^k B_i$, for $k = 1, \ldots, n-1$. Then $N_m < \epsilon/\alpha$ on $A_i \Delta Z_i$. Let $h = \sum_{k=1}^n \chi_{Z_k} s_k$. Since $x \in \bigcup_{k=1}^n A_k \Delta Z_k$ when $g(x) \neq h(x)$, we have that $\|g - h\|_{N_m} \leq \epsilon$ and the result follows.

Corollary 5.12 *If $m \in M_t(X, E')$ and $f \in E^X$, then f is (VR)-integrable with respect to m iff it is (VR)-integrable with respect to $\bar{m}$.In this case we have $(VR) \int f dm = (VR) \int f d\bar{m}$.*

Theorem 5.13 *For $m \in M_t(X, E')$ and $f \in E^X$ the following are equivalent:*
(1) f is (VR)-integrable with respect to m.
(2) For each $\epsilon > 0$, $f|X_{m,\epsilon}$ is continuous and the set $D = \{x : \|f(x)\| N_m(x) \geq \epsilon\}$ is τ_m-compact.

Proof: (1) $\Rightarrow$ (2). Choose $g \in S(X, E)$ such that $\|f - g\|_{N_m} < \epsilon^2$. Let $x_o \in X_{m,\epsilon}$ and $V = \{x : \|g(x) - g(x_o)\| < \epsilon\}$. If $x \in V \cap X_{m,\epsilon}$, then $\|f(x) - g(x)\| \leq \epsilon$ and so $\|f(x) - f(x_o)\| \leq \epsilon$, which proves that $f|X_{m,\epsilon}$ is continuous. To prove that D is τ_m-compact, choose $g \in S(X, E)$ with $\|g - f\|_{N_m} < \epsilon$. Then

$$\{x : \|f(x)\| N_m(x) \geq \epsilon\} = \{x : \|g(x)\| N_m(x) \geq \epsilon\}.$$

Let $A_1, \ldots, A_n \in K(X)$ be disjoint and s_i non-zero elements of E such that $g = \sum_{k=1}^n \chi_{A_k} s_k$. Then

$$A_k \bigcap \{x : \|g(x)\| N_m(x) \geq \epsilon\} = \{x : \|s_k\| N_m(x) \geq \epsilon\} = A_k \bigcap \{x : N_m(x) \geq \epsilon/\|s_k\|\} = D_k.$$

Thus $D = \bigcup D_k$ is τ_m-compact.
(2) $\Rightarrow$ (1). Our hypothesis implies (in view of Proposition 5.8) that f is τ_m-continuous. Since D is τ_m-compact and N_m is τ_m-upper semicontinuous, there exists a positive number α such that $N_m(x) < \alpha$ for each $x \in D$. For each $x \in D$, the set $M_x = \{y : \|f(y) - f(x)\| < \epsilon/\alpha\}$ is a τ_m-clopen neighborhood of x. If $M_x \cap M_y \neq \emptyset$, then $M_x = M_y$. Hence there are $a_1, \ldots, a_n \in D$ such that the sets M_{a_k} are disjoint and cover D. Let $0 < \epsilon_1 < \alpha$ be such that $\|f(a_k\|\epsilon_1 < \epsilon$, for $k = 1, \ldots, n$. There are $A_k \in K(X)$ such that $M_{a_k} \cap Y = A_k \cap Y$, where $Y = \{x : N_m(x) \geq \epsilon_1\}$. Take $Z_1 = A_1, Z_{k+1} = B_{k+1} \setminus \bigcup_{i=1}^k A_i$, for $k = 1, \ldots, n-1$. Then $Z_k \cap Y = A_k \cap Y$. Let $g = \sum_{k=1}^n \chi_{A_k} f(a_k)$. Then $\|f(x) - g(x)\| N_m(x) \leq \epsilon$ for all x. To show this, we consider the two possible cases. Case I: $x \in D$. Then $x \in M_{a_k}$, for some k,

and so $\|f(x) - f(a_k)\|N_m(x) \leq \alpha\|f(x) - f(a_k)\| < \epsilon$. Since $\|f(x)\|N_m(x) \geq \epsilon$, we have $\|f(x)\| = \|f(a_k)\|$. If now $x \in Y$, then $x \in Z_k$ and so $g(x) = f(a_k)$, which implies that $\|f(x) - g(x)\|N_m(x) = \|f(x) - f(a_k)\|N_m(x) < \epsilon$. If $x \notin Y$, then $\|f(x)\|N_m(x) = \|f(a_k)\|N_m(x) \leq \epsilon_1\|f(a_k)\| < \epsilon$, a contradiction.
Case II: $x \notin D$. Then $\|f(x)\|N_m(x) < \epsilon$. If $\|f(x) - g(x)\|N_m(x) > \epsilon$, then $\|g(x)\|N_m(x) > \epsilon$ and so $x \in Z_k$, for some k, which implies that $g(x) = f(a_k)$ and so $\|f(a_k)\|N_m(x) > \epsilon$. Consequently, $N_m(x) > \epsilon_1$ and thus $x \in Z_k \cap Y = M_{a_k} \cap Y$. But then

$$\|f(x) - g(x)\|N_m(x) = \|f(x) - f(a_k)\|N_m(x) < \epsilon_1\epsilon/\alpha < \epsilon,$$

a contradiction. Thus $\|f - g\|_{N_m} \leq \epsilon$ which proves that f is (VR)-integrable with respect to m and we are done.

Lemma 5.14 *If $\phi \in E'$ and Y a compact subset of X, then there exists an $m \in M_t(X, E')$ such that $N_m(x) = \|\phi\|$ for $x \in Y$ and $N_m(x) = 0$ for $x \notin Y$.*

Proof: By [16], p. 273, thete exists a $\mu \in M_\tau(X)$ such that $N_\mu(x) = 1$ for $x \in Y$ and $N_\mu(x) = 0$ for $x \notin Y$. Let $m : K(X) \to E', m(A) = \mu(A)\phi$. Then $m \in M_t(X, E')$ and $N_m = \|\phi\|N_\mu$, which proves the Lemma.

Theorem 5.15 *If $f \in C_{b,k}(X, E)$, then f is (VR)-integrable with respect to every $m \in M_t(X, E')$. If E is polar, then the converse is also true.*

Proof: Assume that $f \in C_{b,k}(X, E)$ and let $m \in M_t(X, E')$. Let $\alpha > \|f|$ and $\epsilon > 0$. Then

$$D = \{x : \|f(x)\|N_m(x) \geq \epsilon\} \subset \{x : N_m(x) \geq \epsilon/\alpha\} = Z.$$

The set Z is τ_m-compact. Also, f is τ_m-continuous (by Theorem 5.13) and N_m is τ_m-upper semicontinuous. Thus D is a τ_m-closed subset of Z and hence D is τ_m-compact. Hence f is (VR)-integrable by Theorem 5.13.
Conversely, assume that E is polar and that the condition is satisfied. We show first that f is bounded. Assume the contrary. Since E is polar, there exists $\phi \in E'$ such that $\sup_{x\in X} |\phi(f(x))| = \infty$. Let $|\lambda| > 1$ and choose a sequence (a_n) of distinct elements of X such that $|\phi(a_n)| > |\lambda|^{2n}$ for all n. Define $m : K(X) \to E', m(A) = (\sum_{a_n\in A})\phi$. Then $m \in M_t(X, E')$. Let $a_n \in A \in K(X)$. If k is the smallest integer with $a_k \in A$, then, for $\phi(s) \neq 0$, we have

$$|m(A)s| = |\sum_{a_i\in A} \lambda^{-i}\phi(s)| = |\lambda^{-k}\phi(s)| \geq |\lambda^{-n}\phi(s)|,$$

and so $|m|(A) \geq |\lambda^{-n}|\|\phi\|$. On the other hand, suppose that $a_n \in A \in K(X)$. There exists a clopen neighborhood B of a_n contained in A and not containing any a_k for $k < n$. If now D is a clopen subset of B, then $|m(D)s| \leq |\lambda^{-n}\phi(s)|$ and so $N_m(a_n) \leq |m|(B) \leq |\lambda^{-n}|\|\phi\|$. Thus $N_m(a_n) = |\lambda^{-n}|\|\phi\|$. But then

$$\|f\|_{N_m} \geq \sup_n \|f(a_n)\|\|\phi\||\lambda|^{-n} \geq \sup_n |\lambda|^{-n}|\phi(f(a_n)| = \infty,$$

a contradiction since f is (VR)-integrable. Thus f is bounded. Let next Y be a compact subset of X and let ϕ be a nonzero element of E'. By the preceding Lemma, there exists an $m \in M_t(X, E')$ such that $N_m(x) = \|\phi\|$ for $x \in Y$ and $N_m(x) = 0$ for $x \notin Y$. Given $\epsilon > 0$, there exists $g \in S(X, E)$ such that $\|f - g\|_{N_m} < \|\phi\|\epsilon$. Let $x_o \in Y$ and $V = \{x : \|g(x) - g(x_o)\| < \|\phi\|\epsilon\}$. If $x \in V \cap Y$, then

$$\|f(x) - f(x_o)\| \leq \max\{\|f(x) - g(x)\|, \|g(x) - g(x_o)\|, \|g(x_o) - f(x_o)\|\} \leq \epsilon,$$

which proves that $f|Y$ is continuous. This completes the proof.

Theorem 5.16 *Let $m \in M_t(X, E')$. If $f \in E^X$ is bounded and m-integrable, then $|\int f dm| \leq \|f\|_{N_m}$.*

Proof: Let $\epsilon > 0$. There exists a clopen partition $A_1, \ldots, A_n$ of X such that , for any clopen partition $D_1, \ldots, D_n$ of X which is a refinement of $A_1, \ldots, A_n$ and any $y_i \in D_i$, we have that $| \int f dm - \sum_{i=1}^{N} m(D_i) f(y_i)| < \epsilon$. Let $\epsilon_1 > 0$ be such that $\|f\|\epsilon_1 < \epsilon$. Choose $x_k \in A_k$ such that $\sup_{x \in A_k} N_m(x) < N_m(x_k) + \epsilon_1$. Now

$$|\int f dm - \sum_{k=1}^{n} m(A_k) f(x_k)| < \epsilon.$$

Moreover

$$|m(A_k) f(x_k)| \leq |m|(A_k)\|f(x_k)\| = [\sup_{y \in A_k} N_m(y)]\|f(x_k)\| \leq [\epsilon_1 + N_m(x_k)]\|f(x_k)\| \leq \epsilon + N_m(x_k)\|f(x_k)\|.$$

Thus

$$|\int f dm| \leq \max\{\epsilon, \max_k |m(A_k) f(x_k)|\} \leq \max\{\epsilon, \epsilon + \sup_{x \in X} N_m(x)\|f(x)\|\}.$$

Taking $\epsilon \to 0$, we get our result.

Theorem 5.17 *Let $m \in M_t(X, E')$ and $f \in E^X$ a bounded function. If f is both integrable and (VR)-integrable with respect to m, then $\int f dm = (VR) \int f dm$.*

Proof: There exists a sequence (g_n) in $S(X, E)$ such that $\|f - g_n\|_{N_m} \to 0$. Since $f - g_n$ is m-integrable and bounded, we have

$$|\int f dm - \int g_n dm| \leq \|f - g_n\|_{N_m} \to 0.$$

Thus,

$$\int f dm = \lim \int g_n dm = (VR) \int f dm.$$

Theorem 5.18 *Let $m \in M_t(X, E')$. For a bounded $f \in E^X$, the following are equivalent:*
(1)f is (VR)-integrable with respect to m.
(2) For every $\epsilon > 0$, $f|X_{m,\epsilon}$ is continuous.
(3) f is τ_m-continuous.
(4) f is (VR)-integrable with respect to $\bar{m}$.
In each of the above cases, we have

$$(VR) \int f dm = (VR) \int f d\bar{m} = \int f dm.$$

Proof: (2) is equivalent to (3) and (1) is equivalent to (4) by Proposition 5.8 and Corollary 5.12. Also (1) implies (2) by Theorem 5.13. Finally, assume that (2) holds and let $d > \|f\|$. Then

$$D = \{x : \|f(x)\|N_m(x) \geq \epsilon\} \subset \{x : N_m(x) \geq \epsilon/d\} = Z.$$

Since f is τ_m-continuous and N_m τ_m-upper semicontinuous, it follows that D is a τ_m-closed subset of the τ_m-compact Z and hence it is τ_m-compact. By Threorem 5.13, f is (VR)-integrable with respect to m. In each of the above cases f is τ_m-continuous and so it is m-integrable and thus

$$(VR)\int f dm = (VR)\int f d\bar{m} = \int f dm$$

by Corollary 5.12 and Theorem 5.17. This completes the proof.

6 Q-Integrals

Theorem 6.1 *Let $m \in M(X, E')$ and $f \in E^X$. Then f is m-integrable iff the following condition is satisfied: For each $\epsilon > 0$, there exists a clopen partition $\{A_1, \ldots, A_n\}$ of X such that, for every x, y which are in the same A_k and any clopen subset B of A_k we have $|m(B)(f(x) - f(y))| \leq \epsilon$.*

Proof: Assume that f is m-integrable and let $\epsilon > 0$. There exists a clopen partition $\{A_1, \ldots, A_n\}$ of X such that, for every clopen partition $\{D_1, \ldots, D_N\}$ of X which is a refinement of $\{A_1, \ldots, A_n\}$ and any choice of $x_k \in D_k$ we have that $|\int f dm - \sum_{k=1}^{N} m(D_k)f(x_k)| \leq \epsilon$. Let now x, y be in some A_i and let B be a clopen subset of A_i. We will show that $|m(B)(f(x) - f(y))| \leq \epsilon$. To prove this, we consider the three possible cases:
Case I. $x, y \in B$. Then it is clear that $|m(B)(f(x) - f(y))| \leq \epsilon$.
Case II. $x, y \in D = A_i \setminus B$. Assume, by way of contradiction, that $|m(B)(f(x) - f(y))| > \epsilon$. Since $\epsilon \geq |m(A_i)(f(x) - f(y))| = |m(B)(f(x) - f(y)) + m(D)(f(x) - f(y))|$, we would have that $|m(B)(f(x) - f(y))| = |m(D)(f(x) - f(y))| \leq \epsilon$, a contradiction.
Case III. $x \in B$ and $y \in D$ (say). Then $|m(A_i)f(y) - [m(B)f(x) + m(D)f(y)]| \leq \epsilon$, i.e. $|m(B)(f(x) - f(y))| \leq \epsilon$.
Thus the condition is satisfied. Conversely, suppose that the condition holds and let $\epsilon > 0$. Let $\{A_1, \ldots, A_n\}$ be as in the condition and let $x_k \in A_k$. If $\{B_1, \ldots, B_N\}$ is a clopen partition of X which is a refinement of $\{A_1, \ldots, A_n\}$ and if $y_j \in B_j$, then for $B_j \subset A_k$, we have that $|m(B_j)[f(y_j) - f(x_k)]| \leq \epsilon$, and thus $|\sum_{k=1}^{n} m(A_k)f(x_k) - \sum_{j=1}^{N} m(B_j)f(y_j)| \leq \epsilon$. This clearly proves that f is m-integrable and hence the result follows.

Let now $m \in M_\tau(X, E')$ and $f \in E^X$. We define $Q_{m,f}$ on X by

$$Q_{m,f}(x) = \inf_{x \in A \in K(X)} \sup\{|m(B)f(x)| : B \subset A, B \in K(X)\}.$$

Also, for $A \in K(X)$, we define

$$\|f\|_{A,Q_m} = \sup_{x \in A} Q_{m,f}(x), \quad \|f\|_{Q_m} = \|f\|_{X,Q_m}.$$

Lemma 6.2 *If* $g = \sum_{k=1}^{n} \chi_{A_k} s_k$, *where* $A_k \in K(X), s_k \in E$, *then* $|\sum_{k=1}^{n} m(A_k)s_k| \leq \|g\|_{Q_m}$.

Proof: We may assume that the A_k are pairwise disjoint. We prove first that, for $A \in K(X), s \in E, h = \chi_A s$, we have that $|m(A)s| \leq \sup_{x \in A} Q_{m,h}(x)$. Indeed, let $\theta > \sup_{x \in A} Q_{m,h}(x)$. For each $x \in A$, there exists a clopen neighborhood V_x of x contained in A such that $|m(B)h(x)| = |m(B)s| < \theta$ for evey clopen set B contained in V_x. Let $\mu = ms$ be defined by $\mu(B) = m(B)s, B \in K(X)$. Then $\mu \in M_\tau(X)$. Since $|\mu|(V_x) < \theta$ for every $x \in A$, it follows that $|\mu|(A) \leq \theta$. Thus $|m(A)s| \leq \theta$, which proves that $|m(A)s| \leq sup_{x \in A} Q_{m,h}(x)$. If $h_k = \chi_{A_k} s_k$, then for $x \in A_k$ we have $Q_{m,h_k}(x) = Q_{m,g}(x)$ and so $|m(A_k)s_k| \leq sup_{x \in A_k} Q_{m,g}(x)$ which clearly completes the proof.

As we have shown in the proof of Theorem 6.1, we have the following

Theorem 6.3 *Let* $m \in M_\tau(X, E')$ *and let* $f \in E^X$ *be m-integrable. Then, given* $\epsilon > 0$, *there exists a clopen partition* $\{A_1, \dots, A_n\}$ *of* X *such that for any* $x_k \in A_k$ *and* $g = \sum_{k=1}^{n} \chi_{A_k} f(x_k)$ *we have that* $|\int f dm - \sum_{k=1}^{n} m(A_k) f(x_k)| \leq \epsilon$ *and* $\|f - g\|_{Q_m} \leq \epsilon$.

Lemma 6.4 *Let* $m \in M_\tau(X, E')$ *and let* $p \in cs(E)$ *be such that* $m_p(X) < \infty$. *If* $f \in E^X$ *is bounded, then* $\|f\|_{Q_m} \leq \|f\|_p m_p(X)$.

Proof: It folows from the fact that, for $B \in K(X)$, we have $|m(B)f(x)| \leq m_p(X)p(f(x))$.

Lemma 6.5 *Let* $m \in M_\tau(X, E')$ *and let* $f \in E^X$ *be m-integrable. Then* $\|f\|_{Q_m} < \infty$.

Proof: There exists $g \in S(X)$ such that $\|f - g\|_{Q_m} \leq 1$. Let $p \in cs(E)$ be such that $m_p(X) \leq 1$. Then

$$\|f\|_{Q_m} \leq \max\{1, \|g\|_{Q_m}\} \leq \max\{1, m_p(X)\|g\|_p\}.$$

Lemma 6.6 *Let* $m \in M_\tau(X, E')$. *If* $f \in E^X$ *is m-integrable, then* $|\int f dm| \leq \|f\|_{Q_m}$.

Proof: Given $\epsilon > 0$, let $\{A_1, \dots, A_n\}$ be a clopen plartition of X such that, for every clopen partition $\{D_1, \dots, D_N\}$ of X which is a refinement of $\{A_1, \dots, A_n\}$ and any choice of $x_k \in D_k$ we have that $|\int f dm - \sum_{k=1}^{N} m(D_k) f(x_k)| \leq \epsilon$. Let $x_k \in A_k$ and $g = \sum_{k=1}^{n} \chi_{A_k} f(x_k)$. Let $x \in A_k$. There exist a clopen subset D of A_k with $x \in D$ such that $|m(B)f(x)| < Q_{m,f}(x) + \epsilon$ for every clopen set $B \subset D$. Thus, for $B \subset D$, we have

$$|m(B)g(x)| = |m(B)f(x_k)| \leq \max\{|m(B)(f(x_k) - f(x))|, |m(B)f(x)|\} \leq Q_{m,f}(x) + \epsilon$$

and so $Q_{m,g}(x) \leq Q_{m,f}(x) + \epsilon$. Now

$$|\int f dm| \leq \max\{\epsilon, |\sum_{k=1}^{n} m(A_k) f(x_k)|\} \leq \max\{\epsilon, \sup_x Q_{m,g}(x)\} \leq \sup_{x \in X} Q_{m,p}(x) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the result follows.

Lemma 6.7 *Let $m \in M_\tau(X, E')$ and $f \in E^X$. If $(g_n) \subset S(X)$ is such that $\|f - g_n\|_{Q_m} \to 0$, then the $\lim_{n\to\infty} \int g_n dm$ exists. Moreover, if (h_n) is another sequence in $S(X)$ such that $\|f - h_n\|_{Q_m} \to 0$, then $\lim_{n\to\infty} \int g_n dm = \lim_{n\to\infty} \int h_n dm$.*

Proof: Since $|\int g_n dm - \int g_k dm| \leq \|g_n - g_k\|_{Q_m} \leq \max\{\|g_n - f\|_{Q_m}, \|f - g_k\|_{Q_m}\}$, it follows that the $\lim_{n\to\infty} \int g_n dm$ exists. If (h_n) is another sequence in $S(X)$ such that $\|f - h_n\|_{Q_m} \to 0$, then

$$|\int g_n dm - \int h_n dm| \leq \max\{\|g_n - f\|_{Q_m}, \|f - h_n\|_{Q_m}\} \to 0.$$

Thus the result follows.

Definition 6.8 *Let $m \in M_\tau(X, E')$. A function $f \in E^X$ is said to be Q-integrable with respect to m if there exists a sequence (g_n) in $S(X)$ such that $\|f - g_n\|_{Q_m} \to 0$. In this case, the $\lim_{n\to\infty} \int g_n dm$ is called the Q-integral of f and will be denoted by $(Q) \int f dm$.*

By what we have shown above, if $f \in E^X$ is m-integrable for some $m \in M_\tau(X, E')$, then f is Q-integrable and $\int f dm = (Q) \int f dm$.

Theorem 6.9 *If $m \in M_t(X, E')$, then every $f \in E^X$ which is (VR)-integrable with respect to m, is also Q-integrable and $(VR) \int f dm = (Q) \int f dm$.*

Proof: It follows from the fact that, if $m \in M_{t,p}(X, E')$, then for each $h \in E^X$ we have $Q_{m,h}(x) \leq N_{m,p}(x) p(h(x))$ for every $x \in X$.

Theorem 6.10 *Assume that E is polar and let $f \in E^X$. If f is Q-integrable with respect to m for each $m \in M_\tau(X, E')$, then f is bounded.*

Proof: Assume that f is not bounded. Since E is polar, there exists $\phi \in E'$ with $\sup_{x\in X} |\phi(f(x))| = \infty$. Let $|\lambda| > 1$ and choose a sequence (a_n) of distinct elements of X such that $|\phi(f(a_n)) > |\lambda|^{2n}$ for all n. Let $m : K(X) \to E', m(A) = (\sum_{a_n \in A} \lambda^{-n})\phi$. Then $m \in M_\tau(X, E')$. Let now $a_n \in A \in K(X)$ and let D be a clopen subset of A containing a_n and not containing any a_k for $k < n$. Then

$$|m(D) f(a_n)| = |(\sum_{a_k \in D} \lambda^{-k})\phi(f(a_n))| = |\lambda|^{-n} |\phi(f(a_n))| \geq |\lambda|^n.$$

This proves that $Q_{m,f}(a_n) \geq |\lambda|^n$ and thus $\|f\|_{Q_m} = \infty$, which implies that f is not Q-integrable with respect to m (in view of Lemma 6.5). This contradiction completes the proof.

For an $m \in M_\tau(X, E')$, define q_m on $C_b(X, E)$ by $q_m(f) = \|f\|_{Q_m}$.

Theorem 6.11 *If* $m \in M_\tau(X, E')$, *then* q_m *is* β*-continuous.*

Proof: It is easy to see that q_m is a non-Archimedean seminorm on $C_b(X, E)$. To prove that q_m is β_o-continuous, let $G \in \Omega$.There exists a decreasing net (A_δ) of clopen subsets of X such that $G = \bigcap \bar{A_\delta}^{\beta_o X}$. Let $p \in cs(E)$ be such that $m_p(X) < \infty$ and $m_p(A_\delta) \to 0$. Let $r > 0$ and choose δ such that $m_p(A_\delta) < 1/r$. The closure in $\beta_o X$ of the set $X \setminus A_\delta$ is disjoint from G. Now

$$V = \{f \in C_b(X, E) : \|f\|_p \leq r, \|f\|_{B,p} \leq 1/m_p(X)\} \subset \{f \in C_b(X, E) : q_m(f) \leq 1\}.$$

Indeed, let $f \in V$. If $x \in A_\delta$, then $Q_{m,f}(x) \leq m_p(A_\delta)p(f(x)) \leq 1$. Also, for $x \in B$ and $D \subset B$, we have $|m(D)f(x)| \leq m_p(X)p(f(x)) \leq 1$ and thus $\|f\|_{Q_m} \leq 1$. This proves that the set $W = \{f \in C_b(X, E) : q_m(f) \leq 1\}$ is a β_G-neighborhood of zero for each $G \in \Omega$ and hence it is a β-neighborhood. Thus q_m is β- continuous.

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SHARP CONDITIONS FOR NONOSCILLATION OF FUNCTIONAL EQUATIONS [1]

J.H. SHEN [1]
Department of Mathematics, Hunan Normal University
Changsha, Hunan 410081, China
E-mail: jhsh@public.cs.hn.cn

I.P. STAVROULAKIS
Department of Mathematics, University of Ioannina
451 10 Ioannina, Greece
E-mail: ipstav@cc.uoi.gr

ABSTRACT

Consider the second order linear functional equation

$$x(g(t)) = P(t)x(t) + Q(t)x(g^2(t)), \qquad (*)$$

where $P, Q \in C([0,\infty),[0,\infty)), g \in C([0,\infty), R), g(t)$ is increasing, $g(t) > t$ or $g(t) < t$ and $g(t) \to \infty$ as $t \to \infty$, and the linear functional equation

$$x(t) - px(t-\tau) + q(t)x(t-\sigma) = 0, \qquad (**)$$

where $p, \tau, \sigma \in (0,\infty), q(t) \in C([0,\infty),[0,\infty))$. We establish the following "sharp" nonoscillation criteria for Eq. $(*)$ and Eq. $(**)$:
Theorem 1. *If $Q(t)P(g(t)) \le 1/4$ for large t, then Eq. $(*)$ has a nonoscillatory solution.*
Theorem 2. *If $\sigma > \tau$ and for large t*

$$p^{-\sigma/\tau} \cdot q(t) \le \left(\frac{\sigma-\tau}{\sigma}\right)^{\sigma/\tau} \cdot \left(\frac{\sigma-\tau}{\tau}\right)^{-1},$$

then Eq. $(**)$ has a nonoscillatory solution.

Keywords. Nonscillation; oscillation; functional equation.

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1. INTRODUCTION

The oscillatory properties of solutions of differential equations with deviating arguments and difference equations with discrete arguments have been the subject of many recent investigations. See, for example, [1,3-5,7,9-11,13,14,18] and the references cited therein. For the oscillatory properties of solutions of functional equations which include difference equations with continuous arguments, the reader is referred to [2,6,12,15,17,19-22].

In 1992, Ladas, Pakula and Wang [12] considered the difference equation

$$x(t) + p_1 x(t-\tau_1) + p_2 x(t-\tau_2) = 0, \quad p_1, p_2, \tau_1, \tau_2 \in R \tag{1.1}$$

and proved that every continuous solution of Eq. (1.1) oscillates if and only if the characteristic equation

$$1 + p_1 e^{-\lambda\tau_1} + p_2 e^{-\lambda\tau_2} = 0 \tag{1.2}$$

has no real roots. Observe that when $p_1, p_2 \in (0,\infty)$, every solution of Eq. (1.1) oscillates. Without loss of generality, it can be assumed that $\tau_1 > \tau_2 > 0$. But then $p_1 > 0$ is a necessary condition for all solutions of Eq. (1.1) to oscillate. On the basis of this discussion they studied the equation

$$x(t) - px(t-\tau) + qx(t-\sigma) = 0, \tag{1.3}$$

where

$$p, q, \tau, \sigma \in (0,\infty) \quad \text{and} \quad \tau < \sigma,$$

and derived the following necessary and sufficient oscillation condition

$$q^\tau \sigma^\sigma > p^\sigma \tau^\tau (\sigma-\tau)^{\sigma-\tau}. \tag{1.4}$$

In 1993 Domshlak [2], in 1995 Zhang and Yan [20], in 1996 Shen [17], in 1997 Zhang, Yan and Zhao [22] and in 1998 Zhang, Yan and Choi [21] studied such equations with variable coefficients, while in 1999, Yan and Zhang [19] considered a system of delay difference equations with constant coefficients. Here, we mention the paper [22] in which the authors considered the difference equation with a variable coefficient of the form

$$x(t) - x(t-\tau) + q(t)x(t-\sigma) = 0, \tag{1.5}$$

where

$$\tau, \sigma \in (0,\infty), \tau < \sigma \quad \text{and} \quad q(t) \in C([0,\infty),(0,\infty)), \tag{1.6}$$

and proved that all solutions of (1.5) oscillate if

$$\liminf_{t\to\infty} q(t) > \left(\frac{\sigma-\tau}{\sigma}\right)^{\sigma/\tau} \cdot \left(\frac{\sigma-\tau}{\tau}\right)^{-1}, \tag{1.7}$$

and Eq. (1.5) has a nonoscillatory solution if

$$\limsup_{t\to\infty} q(t) < \left(\frac{\sigma-\tau}{\sigma}\right)^{\sigma/\tau} \cdot \left(\frac{\sigma-\tau}{\tau}\right)^{-1}, \tag{1.8}$$

with the additional condition

$$|q(t') - q(t'')| \leq L|t' - t''| \quad \text{for any } t', t'' \in (0, \infty), \tag{1.9}$$

where $L > 0$ is some constant.

In the above mentioned papers the equations under consideration are called difference equations with continuous arguments (or continuous variables or continuous time) most likely because constant time delays appear in these equations.

In 1994, Golda and Werbowski [6] studied the second order linear functional equation of the form

$$x(g(t)) = P(t)x(t) + Q(t)x(g^2(t)), \quad t \geq 0, \tag{1.10}$$

where $P, Q : R^+ \to R^+, g : R^+ \to R$ $(R^+ = [0, \infty))$ are given real valued functions, $g(t) \not\equiv t$ for $t \geq 0, \lim_{t\to\infty} g(t) = \infty$, and g^m denotes the m-th iterate of the function g, i.e.,

$$g^0(t) = t, \quad g^{i+1}(t) = g(g^i(t)), \quad t \geq 0, \quad i = 0, 1, 2, \cdots,$$

and established several oscillation conditions. In particular, they proved that all solutions of Eq. (1.10) oscillate if

$$\liminf_{t\to\infty} Q(t)P(g(t)) > \frac{1}{4}. \tag{1.11}$$

It should be emphasized that condition (1.11) (*resp.* (1.7)) is a "sharp" condition in the sense that, when $P(t) \equiv p > 0, Q(t) \equiv q > 0$ and $g(t) = t - \tau, \tau > 0$ (*resp.* $q(t) \equiv q > 0$), it reduces to

$$pq > \frac{1}{4} \quad \left(resp. \ q > \left(\frac{\sigma - \tau}{\sigma}\right)^{\sigma/\tau} \cdot \left(\frac{\sigma - \tau}{\tau}\right)^{-1} \right), \tag{1.12}$$

which is a necessary and sufficient condition for the oscillation of all solutions of

$$x(t - \tau) = px(t) + qx(t - 2\tau) \quad (resp. \ x(t) - x(t - \tau) + qx(t - \sigma) = 0)$$

because if we consider the last two equations then (1.4) reduces to the two conditions in (1.12) respectively.

Note that all the above mentioned papers deal with the oscillatory behavior only except [22] in which the nonoscillation conditions (1.8) and (1.9) were established for Eq. (1.5).

From the above discussion, the questions naturally arise as to whether the conditions

$$Q(t)P(g(t)) \leq 1/4 \quad \text{for large } t \tag{1.13}$$

and

$$q(t) \leq \left(\frac{\sigma - \tau}{\sigma}\right)^{\sigma/\tau} \cdot \left(\frac{\sigma - \tau}{\tau}\right)^{-1} \quad \text{for large } t \tag{1.14}$$

imply that Eq. (1.10) and (1.5) have a nonoscillatory solution respectively.

The aim of this paper is to give answers to the above questions. We will prove that, under additional conditions on $g(t)$, condition (1.13) implies that Eq. (1.10) has a nonoscillatory solution. We will also prove that (1.14) is sufficient to guarantee the existence of a nonoscillatory solution of Eq. (1.5). It is to be noted that condition (1.9) is no longer required in our result and condition (1.14) is weaker than condition (1.8). The last result is given by considering the more general equation of the form

$$x(t) - px(t-\tau) + q(t)x(t-\sigma) = 0, \tag{1.15}$$

where $p \in (0,\infty)$ and τ, σ and $q(t)$ satisfy (1.6).

By a solution of (1.10) (*resp.* (1.15)) we understand a continuous real valued function $x : R^+ \to R$ such that $\sup\{|x(s)| : s \geq t_0\} > 0$ for any $t_0 \geq 0$ and x satisfies (1.10) (*resp.* (1.15)) on $[0,\infty)$. Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Thus a nonoscillatory solution is either eventually positive or eventually negative.

2. MAIN RESULTS

2.1. Nonoscillation criteria for Eq. (1.10)

We will use the following hypotheses for Eq. (1.10).

(H_1) $P(t) \in C(R^+, (0,\infty)), Q(t) \in C(R^+, R^+)$;

(H_2) $g(t) \in C(R^+, R), g(0) = -r_1 \leq g(t) < t$ (retarded argument), $r_1 > 0, g(t) \to \infty$ as $t \to \infty$ and $g(t)$ is strictly increasing;

(H_3) $g(t) \in C(R^+, (0,\infty)), g(t) > t$ (advanced argument), and is strictly increasing.

Theorem 2.1. *Let (H_1) holds. Assume that either (H_2) or (H_3) is satisfied. If*

$$Q(t)P(g(t)) \leq 1/4 \quad for\ large\ t, \tag{1.13}$$

then Eq. (1.10) *has a nonoscillatory solution.*

To prove Theorem 2.1, we need the following lemmas.

Lemma 2.1. *Consider the first order nonlinear functional equation*

$$u(t) = \frac{1}{1 - a(t)u(t-1)}, \quad t \geq 0, \tag{2.1}$$

where $a(t) \in C(R^+, R^+)$ is a given function. Assume that

$$a(t) \leq 1/4 \quad for\ large\ t. \tag{2.2}$$

Then Eq. (2.1) *has an eventually positive continuous solution $u(t)$.*

Proof. Without less of generality, we assume that

$$0 \leq a(t) \leq 1/4 \quad \text{for} \quad t \geq 0. \tag{2.3}$$

Set

$$\alpha = \begin{cases} \frac{1-\sqrt{1-4a(0)}}{2a(0)}, & \text{if } a(0) > 0 \\ 1, & \text{if } a(0) = 0. \end{cases} \tag{2.4}$$

Then α satisfies the relation

$$\alpha = \frac{1}{1 - a(0)\alpha}. \tag{2.5}$$

We claim that

$$1 \le \alpha \le 2. \tag{2.6}$$

Indeed, let

$$f(\xi) = \frac{1 - \sqrt{1 - 4\xi}}{2\xi}, \quad 0 < \xi \le 1/4.$$

Then

$$f'(\xi) = \frac{1 - 2\xi - \sqrt{1 - 4\xi}}{2\xi^2 \sqrt{1 - 4\xi}}, \quad 0 < \xi < 1/4.$$

Set

$$F(\xi) = 1 - 2\xi - \sqrt{1 - 4\xi}, \quad 0 \le \xi \le 1/4.$$

Then

$$F'(\xi) = 2\left(\frac{1}{\sqrt{1 - 4\xi}} - 1\right) > 0, \quad 0 < \xi < 1/4.$$

Thus, $F(\xi)$ is strictly increasing on $(0, 1/4)$. Since $F(0) = 0$, it follows that $F(\xi) > 0$ for $0 < \xi \le 1/4$. Therefore, $f'(\xi) > 0$ for $0 < \xi < 1/4$. Noting that $f(1/4) = 2$ and

$$\lim_{\xi \to 0+} f(\xi) = \lim_{\xi \to 0} \frac{1 - \sqrt{1 - 4\xi}}{2\xi} = \lim_{\xi \to 0} \frac{1}{\sqrt{1 - 4\xi}} = 1,$$

we have $1 < f(\xi) \le 2$ for $0 < \xi \le 1/4$. This and (2.4) lead to (2.6).

Next, we define a function $u(t)$ as follows:

$$u(t) = \begin{cases} \frac{1}{1-a(0)\alpha}, & -1 \le t \le 0, \\ \frac{1}{1-a(t)u(t-1)}, & k < t \le k+1, \quad k = 0, 1, 2, \ldots \end{cases} \tag{2.7}$$

From (2.5), it is not difficult to see that

$$\lim_{t \to 0^+} u(t) = \frac{1}{1 - a(0)u(-1)} = \frac{1}{1 - a(0)\alpha} = u(0). \tag{2.8}$$

(2.7) and (2.8) imply that $u(t)$ is continuous on $[-1, \infty)$. We prove that

$$u(t) \ge 1 \quad \text{for} \quad t \ge -1. \tag{2.9}$$

Indeed, from (2.5), (2.6) and (2.7), we have

$$1 \le u(t) \le 2 \quad \text{for} \quad -1 \le t \le 0. \tag{2.10}$$

For $0 < t \leq 1$, by (2.2), (2.7) and (2.9), we have

$$1 \leq u(t) = \frac{1}{1 - a(t)u(t-1)} \leq \frac{1}{1 - 2a(t)} \leq 2.$$

In general we have $1 \leq u(t) \leq 2$ for $k < t \leq k+1, k = 0, 1, 2, \ldots$. Thus, (2.9) holds. From (2.7) we see that

$$u(t) = \frac{1}{1 - a(t)u(t-1)}, \quad t \geq 0.$$

This shows that $u(t)$ is a positive continuous solution of (2.1). The proof is complete.

We now give some notations on the function $g(t)$. If $g(t)$ satisfies the condition (H_2), then $g^{-1}(t)$ $(g^{-1}(t) > t)$ denotes the inverse of the function $g(t)$ and $g^{-k}(t)$ is defined by $g^{-k-1}(t) = g^{-1}(g^{-k}(t)), k = 1, 2, \ldots$; If $g(t)$ satisfies the condition (H_3), then $g_{-1}(t)$ $(g_{-1}(t) < t)$ denotes the inverse of the function $g(t)$ and $g_{-k}(t)$ is defined by $g_{-k-1}(t) = g_{-1}(g_{-k}(t)), k = 1, 2, \ldots$.

Lemma 2.2. *Consider the first order nonlinear functional equation*

$$W(t) = \frac{1}{1 - b(t)W(g(t))}, \quad t \geq 0, \tag{2.11}$$

where $b(t) \in C(R^+, R^+)$ and $g(t)$ satisfies the condition (H_2). Then there exists a continuous change of variables that transforms Eq. (2.11) into Eq. (2.1). Such a change of variables is given by $u(t) = W(h(t)), t \geq 0$, and $a(t) = b(h(t))$, where $h(t)$ is defined by

$$h(t) = g^{-n}(\psi(t-n)), \quad n-1 \leq t \leq n, \quad n = 0, 1, 2, \ldots \tag{2.12}$$

and $\psi : [-1, 0] \to [-r_1, \infty)$ is any continuous increasing function satisfying the condition

$$g(\psi(0^-)) = \psi(-1^+). \tag{2.13}$$

Furthermore, we have that $u(\cdot)$ defined by $u(t) = W(h(t))$ oscillates if and only if $W(\cdot)$ oscillates.

Proof. Replacing t by $h(t)$ in (2.11) we have (cf.[1])

$$W(h(t)) = \frac{1}{1 - b(h(t))W(g(h(t)))}. \tag{2.14}$$

The term on the left side is just $u(t)$. To complete the transformation it suffices to have $a(t) = b(h(t))$ and $g(h(t)) = h(t-1)$, for $t \geq 0$. From (2.12), we have

$$h(t) = \psi(t), \quad -1 \leq t \leq 0,$$

$$h(t) = g^{-1}(h(t-1)), \quad n-1 \leq t \leq n, \quad n = 1, 2, \ldots$$

By (2.13) we see that h is continuous. Since ψ is increasing on $[-1,0]$, it follows that h is increasing. Finally, to see that $u(\cdot)$ oscillates if and only if $W(\cdot)$ oscillates, it suffices to prove that $h(t) \to \infty$ as $t \to \infty$. Indeed, if $u(\cdot)$ oscillates, then there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and $u(t_n) = 0$. Let $s_n = h(t_n)$, then $s_n \to \infty$ as $n \to \infty$ because $h(t) \to \infty$ as $t \to \infty$. Thus, $W(s_n) = W(h(t_n)) = u(t_n) = 0$. This shows that $W(\cdot)$ oscillates. Conversely, if $W(\cdot)$ oscillates, then there exists a sequence $\{s_n\}$ such that $s_n \to \infty$ as $n \to \infty$ and $W(s_n) = 0$. Let $t_n = h^{-1}(s_n)$ (here h^{-1} is the inverse of the function h), then $t_n \to \infty$ as $n \to \infty$ and $u(t_n) = W(h(t_n)) = W(s_n) = 0$. Thus, $u(\cdot)$ oscillates. Now, to prove that $h(t) \to \infty$ as $t \to \infty$, we need only to prove that $h(n) \to \infty$ as $n \to \infty$, where n takes only integer values. Otherwise, the sequence $h(n) = g^{-n}(\psi(0))$ has a limit L, then

$$g^{-1}(L) = g^{-1}\left(\lim_{n\to\infty} g^{-n}(\psi(0))\right) = \lim_{n\to\infty} g^{-(n+1)}(\psi(0)) = L.$$

This is impossible because $g^{-1}(t) > t$ for all t. The proof is complete.

Remark 2.1. One way to transform (2.11) into (2.1) is to suppose that the function ψ has the form $\psi(t) = at + b$, where a and b are to be determined. We first require $\psi(-1) = -r_1$, which gives $b - a = -r_1$. In addition, condition (2.13) requires $g(b) = -a + b$. From (H_2), $g(0) = -r_1$, it follows that $b = 0, a = r_1$. Thus $\psi(t) = r_1 t$.

Lemma 2.3. *Assume that (H_1) and (H_2) hold. Then Eq.* (1.10) *has an eventually positive solution if and only if the first order nonlinear functional equation*

$$W(t) = \frac{1}{1 - Q(t)P(g(t))W(g(t))}, \quad t \geq 0 \tag{2.15}$$

has an eventually positive continuous solution.

Proof. Assume that $x(t)$ is an eventually positive solution of Eq. (1.10). Dividing both sides of (1.10) by $x(g(t))$ gives

$$1 = P(t)\frac{x(t)}{x(g(t))} + Q(t)\frac{x(g^2(t))}{x(g(t))}. \tag{2.16}$$

Set

$$\overline{W}(t) = \frac{x(g(t))}{P(t)x(t)}. \tag{2.17}$$

Then $\overline{W}(t)$ is eventually positive and continuous and satisfies

$$1 = \frac{1}{\overline{W}(t)} + Q(t)P(g(t))\overline{W}(g(t)), \tag{2.18}$$

which shows that $\overline{W}(t)$ is an eventually positive continuous solution of (2.15).

Next assume that $W(t)$ is an eventually positive continuous solution of (2.15). Without loss of generality, we may assume that $W(t) > 0$ for $t \geq 0$. By similar arguments, as in the proof of Lemma 2.2, we see that there exists a continuous change of variables that transforms the equation

$$x(t) = \frac{1}{W(t)P(t)} x(g(t)), \quad t \geq 0 \tag{2.19}$$

into the equation

$$y(t) = R(t)y(t-1), \quad t \geq 0, \tag{2.20}$$

where $R(t) = [W(h(t))P(h(t))]^{-1}, h(t)$ is as in Lemma 2.2, and Eq. (2.19) has an eventually positive continuous solution $x(t)$ if and only if Eq. (2.20) has an eventually positive continuous solution $y(t)$. Since $R(t) > 0$ for $t \geq 0$, it is easy to see that Eq. (2.20) has an eventually positive continuous solution. Indeed, the function $y(t)$ defined by

$$y(t) = \begin{cases} r(t), & -1 \leq t \leq 0, \\ R(t)y(t-1), & k < t \leq k+1, \quad k = 0, 1, 2, \ldots, \end{cases}$$

where $r(t)$ is any positive continuous function on $[-1, 0]$ such that $r(0) = R(0)r(-1)$, is a positive continuous solution of (2.20). Thus, Eq. (2.19) has an eventually positive continuous solution. Let $\overline{x}(t)$ be such a solution. Substituting

$$W(t) = \frac{\overline{x}(g(t))}{\overline{x}(t)P(t)}$$

into (2.15), we obtain

$$\frac{\overline{x}(g(t))}{\overline{x}(t)P(t)} \left(1 - Q(t)P(g(t)) \frac{\overline{x}(g^2(t))}{\overline{x}(g(t))P(g(t))}\right) = 1,$$

i.e.,

$$\overline{x}(g(t)) = P(t)\overline{x}(t) + Q(t)\overline{x}(g^2(t)).$$

This shows that $\overline{x}(t)$ is a positive solution of (1.10). The proof is complete.

Proof of Theorem 2.1. We first consider the case when $g(t)$ satisfies (H_2). By Lemma 2.3, it suffices to prove that Eq. (2.15) has an eventually positive continuous solution. Set $b(t) = Q(t)P(g(t))$ and let $h(t)$ be a change of variables as in Lemma 2.2. Define $a(t) = b(h(t)) = Q(h(t))P(g(h(t)))$ and $u(t) = W(h(t))$. From condition (1.13), we have

$$0 \leq a(t) = Q(h(t))P(g(h(t))) \leq 1/4 \quad \text{for large} \quad t.$$

Thus, by Lemma 2.1, Eq. (2.1) has an eventually positive continuous solution $u(t)$. By Lemma 2.2, we see that Eq. (2.15) has an eventually positive continuous solution $W(t)$.

Next, we consider the case when $g(t)$ satisfies (H_3). Since $g(t)$ satisfies (H_3), it follows that $g_{-1}(t)$ satisfies (H_2), with the possible exception that $g_{-1}(0) = -r_2 < 0, r_2 \neq r_1$. Replacing $g^2(t)$ by t in Eq. (1.10), we have

$$x(g_{-1}(t)) = Q(g_{-2}(t))x(t) + P(g_{-2}(t))x(g_{-2}(t)). \tag{2.21}$$

Condition (1.13) implies that for t sufficiently large

$$0 \leq P(g_{-2}(t))Q(g_{-2}(g_{-1}(t))) = Q(g_{-3}(t))P(g_{-2}(t)) \leq 1/4.$$

As in the case when $g(t)$ satisfies (H_2), we see that Eq. (2.21) has an eventually positive solution. Thus, Eq. (1.10) has an eventually positive solution. The proof is complete.

Example 2.1. Consider the equation

$$x(t/2) = e^t x(t) + \frac{1}{4} e^{-t/2} x(t/4). \tag{2.22}$$

It is easy to see that

$$Q(t)P(g(t)) = \frac{1}{4} e^{-t/2} \cdot e^{t/2} = \frac{1}{4}.$$

Thus, by Theorem 2.1, Eq. (2.22) has a nonoscillatory solution. In fact, $x(t) = t^{-1}e^{-2t}$ is such a nonoscillatory solution.

2.2. Nonoscillation criteria for Eq. (1.15)

We will establish the following nonoscillation theorem for Eq. (1.15).

Theorem 2.2. *Let $\sigma > \tau$. Assume that*

$$p^{-\sigma/\tau} \cdot q(t) \leq \left(\frac{\sigma - \tau}{\sigma}\right)^{\sigma/\tau} \cdot \left(\frac{\sigma - \tau}{\tau}\right)^{-1} \quad for\ large\ t. \tag{2.23}$$

Then Eq. (1.15) *has a nonoscillatory solution.*

Remark 2.2. When $p = 1$, condition (2.23) reduces to condition (1.14) and Eq. (1.15) reduces to Eq. (1.5). Thus condition (1.14) is sufficient for Eq. (1.5) to have a nonoscillatory solution. On the other hand, condition (2.23) is "sharp" in the sense that when $q(t) \equiv q > 0$ condition (2.23) also is necessary for Eq. (1.15) to have a nonoscillatory solution (cf.(1.4)).

To prove this theorem, we need two intermediate results. The first one is Schauder's fixed-point theorem [16].

Lemma 2.4. *Let Ω be a nonempty bounded closed convex subset of a Banach space $(B, ||\cdot||)$, and let $S : \Omega \to \Omega$ be a continuous mapping such that $S(\Omega)$ is (relatively) compact (S is completely continuous). Then, $S(x) = x$, for some $x \in \Omega$.*

The second one is the following version of Ascoli's theorem.

Lemma 2.5. *Let $\{f_n : [0, \infty) \to R, n = 1, 2, ...\}$ be a sequence of functions such that:*

(i). *there exists a constant $M > 0$ such that $|f_n(t)| \leq M$ for all $n \geq 1$ and $t \geq 0$;*

(ii). *$f_n(t)$ is continuous on $[0, \infty)$ for all $n \geq 1$;*

(iii). *there exist constants $L > 0, \mu > 0$ such that $0 \leq f_n(t) \leq Le^{-\mu t}$ for $t \geq T > 0$ and all $n \geq 1$, where T is a constant.*

Then there exists a continuous function $f : [0, \infty) \to R$ and a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n(t) \to f(t)$ as $n \to \infty$, uniformly on $[0, \infty)$.

Remark 2.3. The authors are unaware of a precise reference for Lemma 2.5, but it is a sequence of the results in [8].

Proof of Theorem 2.2. Let the right side of (2.23) be c. Then, by the results in [12] (see also condition (1.4)), the equation

$$u(t) - u(t - \tau) + cu(t - \sigma) = 0 \tag{2.24}$$

has an nonoscillatory solution of the form $u(t) = e^{\lambda t}$, where $\lambda < 0$ is a root of the characteristic equation $1 - e^{-\lambda\tau} + ce^{-\lambda\sigma} = 0$. It is clear that the equality holds:

$$u(t) = c\sum_{i=1}^{\infty} u(t + i\tau - \sigma).$$

For each real number r, let us define B_r as the space of all real bounded continuous functions defined on $[r, \infty)$, provided with the usual sup-norm; and let $\Omega_r := \{v \in B_r : 0 \leq v(t) \leq e^{\lambda t}, t \geq r\}$. It is clear that Ω_r is a nonempty bounded closed convex subset of the Banach space B_r. Let $T > 0$ be such that (2.23) holds for $t \geq T$. Define a mapping S on Ω_T as follows:

$$S(y)(t) = \begin{cases} \sum_{i=1}^{\infty} q^*(t + i\tau)y(t + i\tau - \sigma), & t \geq T + \sigma - \tau, \\ S(y)(T + \sigma - \tau) + u(t) - u(T + \sigma - \tau), & T \leq t < T + \sigma - \tau, \end{cases}$$

where

$$q^*(t) = p^{-\sigma/\tau} \cdot q(t) \leq c, \quad t \geq T.$$

Then for $t \geq T + \sigma - \tau$

$$0 \leq S(y)(t) \leq \sum_{i=1}^{\infty} cu(t + i\tau - \sigma) = u(t) = e^{\lambda t}. \tag{2.25}$$

While for $T \leq t < T + \sigma - \tau$, we also have

$$0 \leq S(y)(t) \leq u(T + \sigma - \tau) + u(t) - u(T + \sigma - \tau) = u(t) \leq e^{\lambda t}.$$

Thus, (2.25) holds for $t \geq T$. For any $y \in \Omega_T$, we claim that $S(y)$ is continuous. Since $\lim_{t\to\infty} u(t) = 0$, it follows that for any $\varepsilon > 0$ there exists $T_1 > T$ such that $u(t) < \varepsilon$ for $t > T_1$. Choose a positive integer N such that $N\tau \geq T_1$. Then for all $t \geq T + \sigma - \tau$ we have

$$\begin{aligned} \sum_{i=m+1}^{n} q^*(t + i\tau)y(t + i\tau - \sigma) &\leq \sum_{i=m+1}^{\infty} cu(t + i\tau - \sigma) \\ &= u(t + m\tau) < \varepsilon \end{aligned}$$

for any $m, n \geq N$, which implies that the series $\sum_{i=1}^{\infty} q^*(t+i\tau)y(t+i\tau-\sigma)$ converges uniformly on $[T + \sigma - \tau, \infty)$. Thus, $S(y)$ is continuous. From this and (2.25), we have $S(\Omega_T) \subset \Omega_T$.

Notice that $0 \leq S(y)(t) \leq e^{\lambda t}$. This and Lemma 2.5 imply that $S(\Omega_T)$ is (relatively) compact. Hence, by Lemma 2.4, $S(y) = y$ for some $y \in \Omega_T$. i.e.,

$$y(t) = \begin{cases} \sum_{i=1}^{\infty} q^*(t+i\tau)y(t+i\tau-\sigma), & t \geq T+\sigma-\tau, \\ y(T+\sigma-\tau)+u(t)-u(T+\sigma-\tau), & T \leq t < T+\sigma-\tau. \end{cases} \tag{2.26}$$

and

$$y(t) - y(t-\tau) + q^*(t)y(t-\sigma) = 0, \quad t \geq T+\sigma. \tag{2.27}$$

We claim that $y(t) > 0$ for $t \geq T$. Since $u'(t) < 0, t \geq T$, from (2.26), we have

$$y(t) > 0, \quad T \leq t < T+\sigma-\tau.$$

Assume that there exists a $t \in [T+\sigma-\tau, \infty)$ such that $y(t) \leq 0$, then we can let

$$t^* = \inf\{t \geq T+\sigma-\tau : y(t) \leq 0\},$$

so that

$$y(t^*) = 0 \quad \text{and} \quad y(t) > 0, \quad T \leq t < t^*.$$

On the other hand, from (2.26), we have

$$\begin{aligned} y(t^*) &= \sum_{i=1}^{\infty} q^*(t^*+i\tau)y(t^*+i\tau-\sigma) \\ &\geq q^*(t^*+\tau)y(t^*+\tau-\sigma) > 0, \end{aligned}$$

a contradiction. Thus $y(t) > 0$ for $t \geq T$. Finally, let us define $x(t) = p^{t/\tau}y(t)$. Then, by (2.27), we have

$$x(t) - px(t-\tau) + q(t)x(t-\sigma) = 0.$$

Thus, $x(t)$ is a nonoscillatory solution of (1.15). The proof is complete.

Example 2.2. Consider the equation

$$x(t) - x(t-1) + \frac{(et-t+1)(2t-11)}{2t(t-1)e^{5.5}}x(t-5.5) = 0.$$

It is not difficult to check that for $t \geq 6$

$$\begin{aligned} q(t) &= \frac{(et-t+1)(2t-11)}{2t(t-1)e^{5.5}} \\ &\leq \left(\frac{5.5-1}{5.5}\right)^{5.5} \cdot (5.5-1)^{-1}. \end{aligned}$$

Thus, by Theorem 2.2, this equation has a nonoscillatory solution. In fact, $x(t) = t^{-1}e^{-t}$ is such a nonoscillatory solution.

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Symmetric Kotz type and Burr multivariate distributions: A maximum entropy characterization

G. Aulogiaris and K. Zografos[1]
Department of Mathematics, University of Ioannina, 45110 Ioannina, GREECE

Abstract

In this paper a maximum entropy characterization is presented for Kotz type symmetric multivariate distributions as well as for multivariate Burr and Pareto type III distributions. Analytical formulae for the Shannon entropy of these multivariate distributions are also derived.

Keywords and phrases: Shannon Entropy, Maximum Entropy Principle, Kotz type multivariate distribution, Burr distribution, Pareto type III distribution.

1 Introduction

The maximum entropy method is a well-known approach to produce the unknown probability density function f, compatible to new information about f in the form of constraints on expected values. Although entropy maximization was first formulated in terms of thermodynamic entropy, the principle of maximum entropy was first introduced as a general method of inference by Jaynes (1957) and it was axiomatically characterized by Shore and Johnson (1980). It has been successfully applied in a remarkable variety of fields and has been also used for the characterization of several standard probability distributions (cf. Kapur (1989), Guiasu (1990), Gzyl (1995)).

Consider a $p-$variate random vector $\mathbf{X}^t = (X_1, ..., X_p)$, with unknown density f. Although f is unknown, suppose that we have access to some information about this density, formulated in terms of a set of information constraints on expected values. Consider the class of $p-$variate density functions $\mathcal{F} = \{f(\mathbf{x}) : E_f[T_i(\mathbf{X})] = \alpha_i, \ i = 0, 1, ..., m\}$, where $T_i, i = 0, 1, ..., m$, are absolutely integrable functions with respect to f and $T_0(\mathbf{x}) = \alpha_0 = 1$. We suppose further that the values of α_i and the form of $T_i, i = 0, 1, ..., m$, are known. The maximum entropy principle suggests to derive the unknown density function of the random vector $\mathbf{X}$, by the model that maximizes the Shannon entropy

$$H(\mathbf{X}) = -\int f(\mathbf{x}) \log f(\mathbf{x}) d\mathbf{x}, \tag{1}$$

subject to the information constraints that define the class $\mathcal{F}$. Jaynes states that the maximum entropy distribution, obtained by this constrained maximization problem, "is

[1] Corresponding author.
E-mail address: kzograf@cc.uoi.gr

the only unbiased assignement we can make; to use any other would amount to arbitrary assumption of information which by hypothesis we do not have." (Jaynes (1957), p. 623).

A considerable part of the literature related to the principle of maximum entropy is devoted to the maximum entropy characterization of the main univariate probability distributions. In the work of Kagan *et al.* (1973), Preda (1982), Bad Dumitrescu (1986), Kapur (1989), Guiasu (1977, 1990), Ebrahimi (2000), Kotz *et al.* (2000) and the references therein, the main univariate probability distributions have been reobtained by maximizing the Shannon entropy, subject to various types of constraints expressed by mean values of random variables. Comparatively little is the literature dealing with the characterization of multivariate distributions by means of the maximum entropy principle. The main reference, from this point of view, is the book of Kapur (1989) which devotes Chapters 4 and 5 for the characterization of some multivariate distributions, and the paper by Zografos (1999) where Pearson's Type II and VII multivariate distributions have reobtained by means of the maximum entropy principle.

In this paper, following Zografos (1999), we will concentrate on the characterization of Kotz type symmetric multivariate distributions as well as Burr and Pareto type III multivariate distributions. Analytical formulae for the Shannon entropy of these multivariate distributions are derived.

2 Symmetric Kotz type multivariate distribution

The $p-$variate random vector $\mathbf{X}^t = (X_1, ..., X_p)$ in R^p is said to have a symmetric Kotz type multivariate distribution, if the density function of $\mathbf{X}$ is defined by

$$f(\mathbf{x}) = C_p \left|\Sigma\right|^{-1/2} \left[(\mathbf{x}-\boldsymbol{\mu})^t\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^{m-1} \exp\left\{-r\left[(\mathbf{x}-\boldsymbol{\mu})^t\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^s\right\}, \qquad (2)$$

for $r, s > 0$, $2m+p > 2$ and C_p a normalizing constant. The normalizing constant C_p is given by

$$C_p = \frac{s\Gamma(p/2)}{\pi^{p/2}\Gamma\left((2m+p-2)/2s\right)} r^{(2m+p-2)/2s}. \qquad (3)$$

The parameter $\boldsymbol{\mu}$ is the mean vector $E(\mathbf{X})$ and the positive definite matrix Σ is related to the variance-covariance matrix of $\mathbf{X}$ (cf. Fang *et al.* (1990), p. 76, 77). When $m=1$, $s=1$ and $r=1/2$, the distribution defined by (2) reduces to a multivariate normal distribution. The above, are particularly appealing family of distributions in constructing models in which the usual normality assumption is not satisfied.

In order to give a maximum entropy characterization of the density function defined by (2) we need the following lemmas. The proof of Lemma 1 is outlined in the Appendix.

Lemma 1. Let R^p is the $p-$dimensional Euclidean space. Then

a) $\int\limits_{R^p} (\mathbf{y}^t\mathbf{y})^{-\mu} \exp[-\kappa(\mathbf{y}^t\mathbf{y})^s]dy = \frac{\pi^{p/2}\Gamma((p-2\mu)/2s)}{s\Gamma(p/2)}\kappa^{(2\mu-p)/2s}, \quad \kappa > 0,\ \mu < \frac{p}{2}.$

b) $\int\limits_{R^p} (\mathbf{y}^t\mathbf{y})^{s-\mu} \exp[-\kappa(\mathbf{y}^t\mathbf{y})^s]dy = \frac{\pi^{p/2}\Gamma(((p-2\mu)/2s)+1)}{s\Gamma(p/2)}\kappa^{((2\mu-p)/2s)-1}, \quad \kappa > 0,\, \mu < s + \frac{p}{2}.$

c) $$\int\limits_{R^p} (\mathbf{y}^t\mathbf{y})^{-\mu} \exp[-\kappa(\mathbf{y}^t\mathbf{y})^s] \log(\mathbf{y}^t\mathbf{y})dy = \frac{\pi^{p/2}\kappa^{(2\mu-p)/2s}}{s^2\Gamma(p/2)} \times [\Gamma'((p-2\mu)/2s) - \log(\kappa)\Gamma((p-2\mu)/2s)],$$

for $\kappa > 0$, $\mu < \frac{p}{2}$. Γ denotes the gamma function and $\Gamma'(t) = (d/dt)\Gamma(t)$.

Lemma 2. For fixed $\alpha > 0$, consider the function $w(x;\alpha)$ of x, defined by

$$w(x;\alpha) = \Psi(x) - \log(\alpha x), \quad \text{for } x > 0,$$

with $\Psi(t) = (d/dt)\log\Gamma(t)$, being the digamma function. The equation $w(x;\alpha) = w(x_0;\alpha)$, has the unique solution $x = x_0$.

Proof. For $x > 0$ consider the function $\varphi(y) = x/(y+x)^2$, $y > 1$. It is obvious that φ is continuous, positive and decreasing in $y > 1$. Hence based on the Cauchy's integral test we have that

$$\sum_{k=1}^{\infty} \varphi(k) - \int_1^{\infty} \varphi(y)dy > 0 \qquad \text{or} \qquad \sum_{k=0}^{\infty} \frac{x}{(k+x)^2} > 1. \tag{4}$$

On the other hand it is well-known that $\frac{d}{dx}\Psi(x) = \sum_{k=0}^{\infty} \frac{1}{(k+x)^2}$. Therefore, based on (4)

$$\frac{d}{dx} w(x;\alpha) = \sum_{k=0}^{\infty} \frac{1}{(k+x)^2} - \frac{1}{x} > 0, \quad x > 0,$$

which means that $w(x;\alpha)$ is strictly increasing in $x > 0$, which completes the proof of the lemma. ■

The following lemma proves that Shannon's entropy, given by (1), is not invariant under linear, non-singular transformations of the random vector $\mathbf{X}$. The proof is immediately obtained from the more general result that the Shannon entropy is not invariant under an invertible transformation of the variables (cf. Darbellay and Vajda (2000)).

Lemma 3. Suppose that $\mathbf{Y}$ is a $p-$variate random vector, $\mathbf{A}$ a non singular square matrix of order p and $\mathbf{u}$ a fixed $p-$dimensional vector. Then

$$H(\mathbf{AY} + \mathbf{u}) = \log|\det(\mathbf{A})| + H(\mathbf{Y}).$$

Theorem 1. Let $\mathbf{Y}^t = (Y_1, ..., Y_p)$ be a $p-$variate random vector in R^p with density g. Let also

$$E_g\left[(\mathbf{Y}^t\mathbf{Y})^s\right] = \frac{2m+p-2}{2sr}, \tag{C1}$$

and

$$E_g[\log(\mathbf{Y}^t\mathbf{Y})] = \frac{1}{s} w \left(\frac{2m+p-2}{2s}; \frac{2sr}{2m+p-2} \right), \tag{C2}$$

for $2m+p>2$, $r,s>0$, and w the function defined in Lemma 2. Then, the unique solution of the maximization problem

$$\max_g H(\mathbf{Y}) = \max_g \left\{ -\int g(\mathbf{y}) \log g(\mathbf{y}) d\mathbf{y} \right\},$$

under the constraints (C1) and (C2), is given by the density

$$g(\mathbf{y}) = C_p(\mathbf{y}^t\mathbf{y})^{m-1} \exp[-r(\mathbf{y}^t\mathbf{y})^s], \quad r,s>0, \quad 2m+p>2,$$

and the normalizing constant C_p is given by (3).

Proof. Based on the Lagrange multipliers method we have

$$H(\mathbf{Y}) - \lambda - \mu\frac{2m+p-2}{2sr} - \kappa\frac{1}{s}w\left(\frac{2m+p-2}{2s}; \frac{2sr}{2m+p-2}\right)$$

$$= \int_{R^p} g(\mathbf{y}) \log[e^{-\lambda}(\mathbf{y}^t\mathbf{y})^{-\kappa} \exp[-\mu(\mathbf{y}^t\mathbf{y})^s] g^{-1}(\mathbf{y})] d\mathbf{y}$$
$$\leq \int_{R^p} g(\mathbf{y}) [e^{-\lambda}(\mathbf{y}^t\mathbf{y})^{-\kappa} \exp[-\mu(\mathbf{y}^t\mathbf{y})^s] g^{-1}(\mathbf{y})] d\mathbf{y} - 1,$$

with equality if and only if

$$g(\mathbf{y}) = e^{-\lambda}(\mathbf{y}^t\mathbf{y})^{-\kappa} \exp[-\mu(\mathbf{y}^t\mathbf{y})^s]. \tag{5}$$

In view of Lemma 1 (a) and the fact that $g(\mathbf{y})$, given by (5), is a density function we have that

$$e^{\lambda} = \frac{\pi^{p/2}\Gamma((p-2\kappa)/2s)}{s\Gamma(p/2)} \mu^{(2\kappa-p)/2s}, \ \mu>0, \ \kappa<\frac{p}{2}. \tag{6}$$

Constraint (C1) and Lemma 1 (b) lead to the relation

$$e^{\lambda}\frac{2m+p-2}{2sr} = \frac{\pi^{p/2}\Gamma(((p-2\kappa)/2s)+1)}{s\Gamma(p/2)} \mu^{((2\kappa-p)/2s)-1}, \mu>0, \ \kappa<\frac{p}{2}.$$

The last equation, taking into account relation (6), gives

$$\mu = \frac{2sr}{2m+p-2}\frac{p-2\kappa}{2s}. \tag{7}$$

On the other hand constraint (C2), Lemma 1 (c) and relation (6), lead to the relation

$$\Gamma\left(\frac{p-2\kappa}{2s}\right) w \left(\frac{2m+p-2}{2s}; \frac{2sr}{2m+p-2}\right) = \Gamma'\left(\frac{p-2\kappa}{2s}\right) - \log(\mu)\Gamma\left(\frac{p-2\kappa}{2s}\right). \tag{8}$$

Based on this last equation, and relations (7), (8) we have that

$$w\left(\frac{2m+p-2}{2s};\frac{2sr}{2m+p-2}\right)=w\left(\frac{p-2\kappa}{2s};\frac{2sr}{2m+p-2}\right).$$

In view of Lemma 2 and the last equation we obtain that $-\kappa = m-1$ and using (7) we obtain that $\mu = r$. Taking into account that $\kappa = 1-m$ and $\mu = r$, it is obtained that $e^{-\lambda} = C_p$, by using relation (6). This completes the proof of the theorem. ■

Based on Theorem 1 and Lemma 3 we can now state a similar characterization result for Kotz type symmetric distribution with density given by (2). In this context, for a $p-$dimensional vector $\boldsymbol{\mu}$ and a positive definite matrix $\boldsymbol{\Sigma}$ of order p, consider the linear transformation $\mathbf{X} = \boldsymbol{\Sigma}^{1/2}\mathbf{Y}+\boldsymbol{\mu}$, of the random vector $\mathbf{Y}$ of Theorem 1. The next corollary states that among all densities f in R^p that satisfy suitable constraints, the Kotz type symmetric distribution with density given by (2) is the unique density that maximizes Shannon's entropy.

Corollary 1. Let $\mathbf{X}^t = (X_1, ..., X_p)$ be a $p-$variate random vector in R^p with density f. Let also

$$E_f\left[(\mathbf{X}-\boldsymbol{\mu})^t\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})\right]^s=\frac{2m+p-2}{2sr}, \qquad \text{(C1*)}$$

and

$$E_f\left[\log\left((\mathbf{X}-\boldsymbol{\mu})^t\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})\right)\right]=\frac{1}{s}w\left(\frac{2m+p-2}{2s};\frac{2sr}{2m+p-2}\right), \qquad \text{(C2*)}$$

for $2m+p>2$, $r,s>0$. Then the unique solution of the maximization problem

$$\max_f H(\mathbf{X}) = \max_f\left\{-\int f(\mathbf{x})\log f(\mathbf{x})d\mathbf{x}\right\},$$

under the constraints (C1*) and (C2*), is the density of the Kotz type symmetric distribution given by (2).

In the next corollary the analytic formula for the entropy of the Kotz type symmetric distribution is presented. The proof can be immediately obtained in view of Corollary 1, relations (1) and (2) and taking into account constraints (C1*) and (C2*) of Corollary 1.

Corollary 2. The Shannon entropy of the Kotz type symmetric distribution with density given by (2) is

$$H(Kotz)=-\log C_p+\frac{1}{2}\log|\boldsymbol{\Sigma}|+r\frac{2m+p-2}{2sr}-(m-1)\frac{1}{s}w\left(\frac{2m+p-2}{2s};\frac{2sr}{2m+p-2}\right),$$

with $C_p = \frac{s\Gamma(p/2)}{\pi^{p/2}\Gamma((2m+p-2)/2s)}r^{(2m+p-2)/2s}$, $w(x;\alpha) = \Psi(x)-\log(\alpha x)$, for $x>0$, and $r,s>0$, $2m+p>2$.

An application of Corollary 2 for $m=1$, $s=1$ and $r=1/2$, leads to the well-known entropy of the multivariate normal distribution that is $\frac{1}{2}\left(p\log(2\pi)+\log|\boldsymbol{\Sigma}|+p\right)$.

3 Burr and Pareto type III multivariate distributions

The aim of this section is to obtain Burr and Pareto type III multivariate distributions by means of the maximum entropy principle. A random vector $\mathbf{X}^t = (X_1, ..., X_p)$ follows a Burr multivariate distribution if the density function of $\mathbf{X}$ is defined by (cf. Johnson and Kotz (1972)),

$$f(\mathbf{x}) = \prod_{i=1}^{p} (\alpha + i - 1) d_i c_i x_i^{c_i - 1} (1 + \sum_{i=1}^{p} d_i x_i^{c_i})^{-(\alpha+p)}, \tag{9}$$

for $x_i > 0$, $c_i > 0$, $d_i > 0$, $i = 1, ..., p$, and $\alpha > 0$. From here and in the sequel we shall be concerned with the case $\alpha = 1$.

The multivariate Pareto type III distribution has density (cf. Johnson and Kotz (1972))

$$f^*(\mathbf{z}) = \prod_{i=1}^{p} \frac{i}{\gamma_i \theta_i} \left(\frac{z_i - \lambda_i}{\theta_i} \right)^{\frac{1}{\gamma_i} - 1} \left(1 + \sum_{i=1}^{p} \left(\frac{z_i - \lambda_i}{\theta_i} \right)^{\frac{1}{\gamma_i}} \right)^{-(1+p)} \tag{10}$$

for $z_i > \lambda_i$, $\gamma_i > 0$, $\theta_i > 0$, $i = 1, ..., p$. If $\alpha = 1$, $c_i = 1/\gamma_i$, and $d_i = 1$, then a Pareto random vector of type III, to be denoted as $\mathbf{Z}$, can be obtained from a Burr random vector $\mathbf{X}$ by the following component-wise transformation

$$z_i = \theta_i x_i + \lambda_i, \quad i = 1, ..., p.$$

Hence the maximum entropy characterization of Pareto type III multivariate distribution can be achieved by the respective one of the multivariate Burr distribution in view of Lemma 3 and the above transformation. In order to present the maximum entropy characterization of Burr distribution we will follow ideas of the previous section. In this context the following lemmas are necessary. The proof of Lemma 4 is outlined in the Appendix.

Lemma 4. *Let* $S = \{\mathbf{y} \in R^p : \ y_i > 0, \ i = 1, ..., p\}$. *Then for* $c_i > 0$, $i = 1, ..., p$,

$$a) \quad \int_S (1 + \sum_{i=1}^{p} y_i^{c_i})^{-\mu} \prod_{i=1}^{p} y_i^{-\kappa_i} d\mathbf{y} = \Gamma(\mu - \sum_{i=1}^{p} \beta_i) \prod_{i=1}^{p} \Gamma(\beta_i)/\Gamma(\mu) \prod_{i=1}^{p} c_i,$$

$$b) \quad \int_S (1 + \sum_{i=1}^{p} y_i^{c_i})^{-\mu} \prod_{i=1}^{p} y_i^{-\kappa_i} \log(1 + \sum_{i=1}^{p} y_i^{c_i}) d\mathbf{y} = (\prod_{i=1}^{p} \Gamma(\beta_i)/\Gamma^2(\mu) \prod_{i=1}^{p} c_i) \times$$
$$\times (\Gamma(\mu - \sum_{i=1}^{p} \beta_i)\Gamma'(\mu) - \Gamma(\mu)\Gamma'(\mu - \sum_{i=1}^{p} \beta_i)),$$

$$c) \quad \int_S \log(y_j)(1 + \sum_{i=1}^{p} y_i^{c_i})^{-\mu} \prod_{i=1}^{p} y_i^{-\kappa_i} d\mathbf{y} = (\prod_{\substack{i=1 \\ i \neq j}}^{p} \Gamma(\beta_i)/c_j \Gamma(\mu) \prod_{i=1}^{p} c_i) \times$$
$$\times (\Gamma(\mu - \sum_{i=1}^{p} \beta_i)\Gamma'(\beta_j) - \Gamma(\beta_j)\Gamma'(\mu - \sum_{i=1}^{p} \beta_i)),$$

for $\beta_i = (1-\kappa_i)/c_i$, with $\beta_i > 0$, κ_i real constants and $\mu > \sum_{i=1}^{p} \beta_i$, $i, j = 1, ..., p$.

Lemma 5. For $p \in N$, consider the system of equations

$$\begin{aligned} \Psi(y) - \Psi(y - px) &= \Psi(1+p) - \Psi(1) \\ \Psi(x) - \Psi(y - px) &= 0 \end{aligned} \quad , \tag{11}$$

for $(x, y) \in V = \{(x, y) \in R^2 : \quad x > 0, \quad y > px\}$, with Ψ the digamma function. The equations (11) have the unique solution $x = 1$ and $y = 1 + p$.

Proof. Taking into account that the digamma function is strictly increasing, the second equation of (11) leads to

$$y = (p+1)x. \tag{12}$$

Hence the first equation of (11) becomes

$$\Psi((p+1)x) - \Psi(x) = \Psi(1+p) - \Psi(1). \tag{13}$$

For $x > 0$ define the function $\omega(x) = \Psi((p+1)x) - \Psi(x)$. It is well known that $\Psi(mx) = \log m + \frac{1}{m} \sum_{\kappa=0}^{m-1} \Psi\left(x + \frac{\kappa}{m}\right)$. The derivative of Ψ with respect to x at the point $m = p+1$ gives that $(p+1)\Psi'((p+1)x) = \frac{1}{p+1} \sum_{\kappa=0}^{p} \Psi'(x + \frac{\kappa}{p+1})$. Taking into account that the function Ψ' is strictly decreasing the last identity leads to $(p+1)\Psi'((p+1)x) < \Psi'(x)$. Hence the function $\omega(x)$, defined above, is strictly decreasing and the equation (13) has the unique solution $x = 1$. From (12) the unique solution with respect to y is $y = 1 + p$ which completes the proof of the lemma. ∎

Theorem 2. Let $\mathbf{Y}^t = (Y_1, ..., Y_p)$ be a $p-$variate random vector in $S = \{\mathbf{y} \in R^p : y_i > 0, i = 1, ..., p\}$ with density g. Let also for $c_i > 0$, $i = 1, ..., p$, the following constraints are satisfied,

$$E_g(\log(1 + \sum_{i=1}^{p} Y_i^{c_i})) = \Psi(1+p) - \Psi(1), \tag{C3}$$

$$E_g(\log(Y_j)) = 0, \quad j = 1, ..., p, \tag{C4}$$

for Ψ the digamma function. In this context, the unique solution of the maximization problem

$$\max_g H(\mathbf{Y}) = \max_g \left\{ - \int_S g(\mathbf{y}) \log g(\mathbf{y}) d\mathbf{y} \right\},$$

under the constraints (C3) and (C4), is given by the density

$$g(\mathbf{y}) = \prod_{i=1}^{p} i c_i y_i^{c_i - 1} (1 + \sum_{i=1}^{p} y_i^{c_i})^{-(1+p)}, \quad c_i > 0, \quad i = 1, ..., p, \text{ and } \mathbf{y} \in S.$$

Proof. Following the steps of the proof of Theorem 1, for constants λ, μ, and κ_i, $i = 1, ..., p$, we have that

$$H(\mathbf{Y}) - \lambda - \mu(\Psi(1+p) - \Psi(1)) \leq \int_S g(\mathbf{y}) \left(e^{-\lambda}(1+\sum_{i=1}^{p} y_i^{c_i})^{-\mu} \prod_{i=1}^{p} y_i^{-\kappa_i} \right) g^{-1}(\mathbf{y}) d\mathbf{y} - 1,$$

with equality if and only if

$$g(\mathbf{y}) = e^{-\lambda}(1+\sum_{i=1}^{p} y_i^{c_i})^{-\mu} \prod_{i=1}^{p} y_i^{-\kappa_i}. \quad (14)$$

From Lemma 4 (a) and the fact that the density $g(\mathbf{y})$ above integrates to 1, we have that

$$e^{\lambda} = \Gamma(\mu - \sum_{i=1}^{p} \beta_i) \prod_{i=1}^{p} \Gamma(\beta_i)/\Gamma(\mu) \prod_{i=1}^{p} c_i, \quad (15)$$

for $\beta_i = (1-\kappa_i)/c_i$, with $\beta_i > 0$, $i = 1, ..., p$, and $\mu > \sum_{i=1}^{p} \beta_i$. Constraint (C3) and Lemma 4 (b) lead, after a little algebra, to the equality

$$\Psi(1+p) - \Psi(1) = \Psi(\mu) - \Psi(\mu - \sum_{i=1}^{p} \beta_i). \quad (16)$$

In a similar manner constraint (C4), Lemma 4 (c) and relation (15) give that

$$\Psi(\beta_j) - \Psi(\mu - \sum_{i=1}^{p} \beta_i) = 0, \quad j = 1, ..., p. \quad (17)$$

Equations (16) and (17) lead,

$$\Psi(\beta_j) = \Psi(\mu) - \Psi(1+p) + \Psi(1), \quad \text{for every } j = 1, ..., p,$$

which means that

$$\Psi(\beta_1) = \Psi(\beta_2) = .. = \Psi(\beta_p).$$

This last equation and the strict monotonicity of digamma function Ψ ensure that

$$\beta_1 = \beta_2 = .. = \beta_p = \beta. \quad (18)$$

Based on them, equations (16) and (17) are equivalent to the following

$$\begin{array}{l} \Psi(1+p) - \Psi(1) = \Psi(\mu) - \Psi(\mu - p\beta), \\ \Psi(\beta) - \Psi(\mu - p\beta) = 0, \end{array} \quad (19)$$

for $\beta > 0$ and $\mu > p\beta$. From Lemma 5 we have that the unique solution of the equations (19) is

$$\beta = 1 \quad \text{and} \quad \mu = 1 + p. \quad (20)$$

It is $\beta_i = (1 - \kappa_i)/c_i$, $i = 1, ..., p$, then from (19) and (20) we have that

$$-\kappa_i = c_i - 1, \quad i = 1, ..., p. \tag{21}$$

Relations (15), (18), and (20) lead to $e^{-\lambda} = \prod_{i=1}^{p} ic_i$, which completes the proof of the theorem in view of (14). ■

The density

$$g(\mathbf{y}) = \prod_{i=1}^{p} ic_i y_i^{c_i - 1}(1 + \sum_{i=1}^{p} y_i^{c_i})^{-(1+p)}, \quad c_i > 0, \ i = 1, ..., p, \tag{22}$$

and $\mathbf{y} \in S = \{\mathbf{y} \in R^p : y_i > 0, \ i = 1, ..., p\}$, obtained in Theorem 2, can be also used in order to generate the multivariate Burr distribution with density given by (9) and the parameter $\alpha = 1$. Indeed, if the $p-$variate random vector $\mathbf{Y}^t = (Y_1, ..., Y_p)$ has density $g(\mathbf{y})$, given by (22), then the random vector $\mathbf{X}^t = (X_1, ..., X_p)$ defined by the following component-wise transformation

$$x_i = d_i^{-1/c_i} y_i, \quad i = 1, ..., p,$$

has a multivariate Burr distribution with density given by (9) and the parameter $\alpha = 1$. This remark associated with Theorem 2 and Lemma 3 lead to the following corollary which states the maximum entropy characterization of Burr multivariate distribution for the parameter $\alpha = 1$.

Corollary 3. Let $\mathbf{X}^t = (X_1, ..., X_p)$ be a $p-$variate random vector in $S = \{\mathbf{x} \in R^p : x_i > 0, \ i = 1, ..., p\}$ with density f. Let also for $c_i > 0$, $i = 1, ..., p$, the following constraints are satisfied,

$$E_f(\log(1 + \sum_{i=1}^{p} d_i X_i^{c_i})) = \Psi(1 + p) - \Psi(1), \tag{C3*}$$

$$E_f(\log(d_j^{1/c_j} X_j)) = 0, \quad j = 1, ..., p, \tag{C4*}$$

for Ψ the digamma function. Then the unique solution of the maximization problem

$$\max_f H(\mathbf{X}) = \max_f \left\{ - \int_S f(\mathbf{x}) \log f(\mathbf{x}) d\mathbf{x} \right\},$$

under the constraints (C3*) and (C4*), is given by the density

$$f(\mathbf{x}) = \prod_{i=1}^{p} id_i c_i x_i^{c_i - 1}(1 + \sum_{i=1}^{p} d_i x_i^{c_i})^{-(1+p)}, \quad c_i > 0, \ d_i > 0, \ i = 1, ..., p, \ \mathbf{x} \in S. \tag{23}$$

Based on the discussion at the beginning of this section, the multivariate Pareto type III distribution with density $f^*(\mathbf{z})$, given by (10), can be generated from (23) for $c_i = 1/\gamma_i$, and $d_i = 1$, and by using the component-wise transformation

$$z_i = \theta_i x_i + \lambda_i, \quad i = 1, ..., p,$$

for $z_i > \lambda_i$, $\gamma_i > 0$, $\theta_i > 0$, $i = 1, ..., p$. This transformation, in association with Corollary 3 and Lemma 3, leads to a similar maximum entropy characterization of the Pareto type III density, given by (10), under the following constraints,

$$E_{f^*}\left(\log\left(1+\sum_{i=1}^{p}\left(\frac{Z_i-\lambda_i}{\theta_i}\right)^{\frac{1}{\gamma_i}}\right)\right) = \Psi(1+p) - \Psi(1), \qquad \text{(C3**)}$$

$$E_{f^*}\left(\log\left(\frac{Z_j-\lambda_j}{\theta_j}\right)\right) = 0, \quad j = 1, ..., p. \qquad \text{(C4**)}$$

The above results can be also used in order to evaluate the Shannon entropy of the multivariate Burr, for $\alpha = 1$, and the multivariate Pareto type III distributions. These entropies are presented in the corollary that follows the proof of which can be immediately obtained in view of Corollary 3, relations (9) and (10) and taking into account constraints (C3*), (C4*) and (C3**), (C4**).

Corollary 4. a) The Shannon entropy of the Burr distribution with density given by (9) and $\alpha = 1$, $c_i > 0$, $d_i > 0$, $i = 1, ..., p$, is

$$H(Burr, \alpha = 1) = -\sum_{i=1}^{p} \log i + (1+p)[\Psi(1+p) - \Psi(1)] - \sum_{i=1}^{p} \log\left(c_i d_i^{1/c_i}\right).$$

b) The Shannon entropy of the Pareto type III distribution, with density given by (10), is

$$H(Pareto\ III) = -\sum_{i=1}^{p} \log\frac{i}{\theta_i} + (1+p)[\Psi(1+p) - \Psi(1)] + \sum_{i=1}^{p} \log(\gamma_i),$$

for $\gamma_i > 0$, $\theta_i > 0$, $i = 1, ..., p$.

The above expressions for the Shannon entropy of Burr and Pareto III multivariate distributions have been also obtained by Darbellay and Vajda (2000) in a different framework.

A Appendix

Proof of Lemma 1. The proof of parts (a) and (c) are given in the sequel. Part (b) can be proved in a similar manner.

(a) Consider the generalized spherical coordinate transformation

$$\begin{aligned} x_1 = & \ r \prod_{k=1}^{p-1} \sin\theta_k \\ x_j = & \ r(\prod_{k=1}^{p-j} \sin\theta_k) \cos\theta_{p-j+1}, \quad 2 \leq j \leq p-1 \\ x_p = & \ r\cos\theta_1 \end{aligned},$$

for $0 < r \leq 1$, $0 < \theta_i \leq \pi$, $i = 1, ..., p-2$ and $0 < \theta_{p-1} \leq 2\pi$. Clearly, we have $\mathbf{x}^t\mathbf{x} = x_1^2 + \cdots + x_p^2 = r^2$ and the Jacobian of the transformation from $x_1, ..., x_p$ to $r, \theta_1, ..., \theta_{p-1}$ is $r^{p-1} \sin^{p-2}\theta_1 \sin^{p-3}\theta_2 \cdots \sin\theta_{p-2}$ (cf. Muirhead (1982), p. 37). Taking into account the equality

$$\int_0^{\pi} \cdots \int_0^{\pi}\int_0^{2\pi} \sin^{p-2}\theta_1 \sin^{p-3}\theta_2 \cdots \sin\theta_{p-2} d\theta_1 \cdots d\theta_{p-2} d\theta_{p-1} = \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})},$$

(cf. Muirhead (1982), p. 37), we have

$$\begin{aligned} \int_{R^p} (\mathbf{y}^t\mathbf{y})^{-\mu} \exp[-\kappa(\mathbf{y}^t\mathbf{y})^s] d\mathbf{y} &= \frac{2\pi^{p/2}}{\Gamma(p/2)} \int_0^{\infty} (r^2)^{-\mu} \exp[-\kappa(r^2)^s]\, r^{p-1} dr \\ &= \frac{\pi^{p/2}}{s\Gamma(p/2)} \int_0^{\infty} z^{\frac{p-2\mu}{2s}-1} \exp(-\kappa z) dz. \end{aligned}$$

The last integral is the gamma function $\Gamma((p-2\mu)/2s)\kappa^{(2\mu-p)/2s}$, $\kappa > 0$, $\mu < \frac{p}{2}$, and the proof of part (a) of the lemma is completed.

(c) If we consider again the generalized spherical coordinate transformation, we have

$$\begin{aligned} \int_{R^p} (\mathbf{y}^t\mathbf{y})^{-\mu} \exp[-\kappa(\mathbf{y}^t\mathbf{y})^s] \log(\mathbf{y}^t\mathbf{y}) d\mathbf{y} &= \frac{2\pi^{p/2}}{\Gamma(p/2)} \int_0^{\infty} (r^2)^{-\mu} \exp[-\kappa(r^2)^s] \log(r^2)\, r^{p-1} dr \\ &= \frac{\pi^{p/2}\kappa^{\frac{p-2\mu}{2s}+1}}{s^2\Gamma(p/2)} \int_0^{\infty} (\kappa z)^{\frac{p-2\mu}{2s}-1} \exp(-\kappa z) \log(z) dz. \end{aligned} \tag{A1}$$

The derivative with respect to t of the gamma function $\Gamma(t) = \int_0^{\infty} x^{t-1} \exp(-x) dx$, $t > 0$, is given by $\Gamma'(t) = \int_0^{\infty} x^{t-1} \exp(-x) \log(x) dx$. For $x = \alpha z$, $\alpha > 0$, we have

$$\Gamma'(t) = \alpha \int_0^{\infty} (\alpha z)^{t-1} \exp(-\alpha z) \log(\alpha) dz + \alpha \int_0^{\infty} (\alpha z)^{t-1} \exp(-\alpha z) \log(z) dz.$$

Hence

$$\alpha \int_0^{\infty} (\alpha z)^{t-1} \exp(-\alpha z) \log(z) dz = \Gamma'(t) - \log(\alpha)\Gamma(t).$$

An application of this last equality, for $\alpha = \kappa$ and $t = (p-2\mu)/2s$, to the relation (A1) leads to the desired result. ■

Proof of Lemma 4. a) Consider the transformation $u_i = y_i^{c_i}$, $i = 1, ..., p$. Then if we denote by $I = \int_S (1+\sum_{i=1}^p y_i^{c_i})^{-\mu} \prod_{i=1}^p y_i^{-\kappa_i} d\mathbf{y}$, we have

$$I = (\prod_{i=1}^p c_i)^{-1} \int_0^\infty \int_0^\infty ... \int_0^\infty \left[\prod_{i=1}^{p-1} u_i^{\beta_i - 1} \left(\int_0^\infty u_p^{\beta_p - 1} (1+\sum_{i=1}^p u_i)^{-\mu} du_p \right) \right] du_1 ... du_{p-1} \quad \text{(A2)}$$

Consider now the integral $I_p = \int_0^\infty u_p^{\beta_p-1}(1+\sum_{i=1}^p u_i)^{-\mu} du_p$. If we will use the transformation $\omega = u_p/(1+\sum_{i=1}^{p-1} u_i)$, then

$$I_p = (1+\sum_{i=1}^{p-1} u_i)^{-\mu+\beta_p} \int_0^\infty \omega^{\beta_p-1}(1+\omega)^{-\mu} d\omega = (1+\sum_{i=1}^{p-1} u_i)^{-\mu+\beta_p} B(\beta_p, \mu-\beta_p),$$

for $\beta_p > 0$, $\mu > \beta_p$ and B the beta function. Taking into account relation (A2)

$$I = (\prod_{i=1}^p c_i)^{-1} B(\beta_p, \mu-\beta_p) \int_0^\infty \int_0^\infty ... \int_0^\infty \left[\prod_{i=1}^{p-1} u_i^{\beta_i-1} \left(1+\sum_{i=1}^{p-1} u_i \right)^{-\mu+\beta_p} \right] du_1 ... du_{p-1}.$$

If the same procedure is repeated $(p-1)$–times then the integral I becomes

$$I = (\prod_{i=1}^p c_i)^{-1} \prod_{i=1}^p B(\beta_i, \mu - \sum_{j=i}^p \beta_j),$$

and leads to the desired result.
b) The proof of this part follows immediately from part a) if we observe that

$$\int_S (1+\sum_{i=1}^p y_i^{c_i})^{-\mu} \prod_{i=1}^p y_i^{-\kappa_i} \log(1+\sum_{i=1}^p y_i^{c_i}) d\mathbf{y} = -\frac{d}{d\mu} \int_S (1+\sum_{i=1}^p y_i^{c_i})^{-\mu} \prod_{i=1}^p y_i^{-\kappa_i} d\mathbf{y}.$$

c) Let $I^* = \int_S \log(y_j)(1+\sum_{i=1}^p y_i^{c_i})^{-\mu} \prod_{i=1}^p y_i^{-\kappa_i} d\mathbf{y}$. Consider the transformation $u_i = y_i^{c_i}$, $i = 1, ..., p$. Then

$$I^* = (c_j \prod_{i=1}^p c_i)^{-1} \int_S \log(u_j)(1+\sum_{i=1}^p u_i)^{-\mu} \prod_{i=1}^p u_i^{\beta_i-1} d\mathbf{u}, \quad \text{(A3)}$$

with $\beta_i = (1-\kappa_i)/c_i$, $i = 1, ..., p$. If we use the transformation $\omega = u_j/(1+\sum_{i=1,i\neq j}^{p-1} u_i)$, and take the derivatives, with respect to β_j, of both sides of the identity $\int_0^\infty \omega^{\beta_j-1}(1+\omega)^{-\mu}d\omega = \Gamma(\beta_j)\Gamma(\mu-\beta_j)/\Gamma(\mu)$, $\beta_j > 0$, $\mu > \beta_j$, after a little algebra, we obtain that

$$\int_0^\infty \log(u_j)u_j^{\beta_j-1}(1+\sum_{i=1}^p u_i)^{-\mu}du_j = (1+\sum_{i=1,i\neq j}^p u_i)^{-\mu+\beta_j}\left(W_1+W_2\log(1+\sum_{i=1,i\neq j}^p u_i)\right), \tag{A4}$$

with $W_1 = [\Gamma'(\beta_j)\Gamma(\mu-\beta_j) - \Gamma(\beta_j)\Gamma'(\mu-\beta_j)]/\Gamma(\mu)$ and $W_2 = \Gamma(\beta_j)\Gamma(\mu-\beta_j)/\Gamma(\mu)$. Using relation (A4) and parts (a) and (b) of Lemma 4, relation (A3) completes the proof of the lemma. ■

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The Steenrod algebra action on Dickson algebra generators and Peterson's polynomials

Nondas E. Kechagias

ABSTRACT. Using Peterson's polynomials, we provide excplicit formulas for the action of the Steenrod algebra on Dickson algebra generators for the mod-odd case.

1. Introduction

The action of the Steenrod algebra on Dickson algebra has been under investigation mainly because it plays an important role in stable homotopy theory and has geometric applications, (see [**2**]). The value of this action on generators has been given for $p = 2$ and partially for p an odd prime.

The main theme of this work is theorem 1 and 2. Those theorems are included in section three, where we investigate the value of the action on the two extreme generators (with respect to degree) of Dickson algebra. Combining theorems 1 and 2 with well known theorems (see [**3**]), the action can be calculated for any generator.

The key point is this action on Peterson's polynomials. We are interested in a special class of Peterson's polynomials called leading Peterson's polynomials. The size of this set is given by a Fibonacci sequence.

In section two, well known results are recollected from the literature for completeness.

2. Dickson and Symmetric invariants

Let $V^{(i)}$ be the i-th dimensional vector space over the field F_p of p-elements generated by $\{y_1, ..., y_i\}$ and GL_i the general linear group acting as usual. Let also GL_n act on the polynomial algebra $P_n := F_p[y_1, ..., y_n]$ by the induced action. P_n is graded by $|y_t| = 2$ (for topological reasons).

Since $P_n \leq H^*(V^{(n)}, F_p)$, the Steenrod algebra acts naturally on.

Let h_i be the polynomial given by .

$$h_i = \prod_{a \in V^{(i-1)}} (y_i + a) \tag{2.1}$$

which has degree $2p^{i-1}$. Let us note that $y_i^{p^{i-1}}$ is a summand in h_i and the last polynomial is invariant under the upper triangular group U_n where only one's are

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allowed on the main diagonal. This is because $g(y_i + a) = y_i + b + a'$, where $b, a' \in V^{(i-1)}$. If non-zero elements are allowed on the diagonal, then h_i^{p-1} is invariant under the Borel subgroup B_n:

$$g(y_i + a)^{p-1} = (cy_i + b + a')^{p-1} = c^{p-1}(y_i + b' + a'')^{p-1}.$$

Since the set $\{h_1^{p-1}, ..., h_n^{p-1}\}$ is algebraically independent and its elements have the right degrees with respect to the order of the group B_n, the corresponding ring of invariants $P_n^{B_n}$ is a polynomial algebra ([5]): $F_p[h_1^{p-1}, ..., h_n^{p-1}]$.

The following proposition is known:

PROPOSITION 1. *Let $f \in P_n$, then it is a GL_n-invariant iff f is symmetric and invariant under the transformation $y_t \to y_t + cy_i$, and $y_i \to y_i$ for $i = 1, ..., t-1$ and $c \in F_p$.*

Let the symmetric group Σ_n act on P_n by permuting variables, and

$$P_n' := F_p[y_1^{p-1}, y_2^{p(p-1)}, ..., y_n^{p^{n-1}(p-1)}]$$

abbreviate the extended polynomial algebra. Then S_n' is called the extended symmetric algebra where:

$$S_n' := F_p[y_1^{p-1}, y_2^{p(p-1)}, ..., y_n^{p^{n-1}(p-1)}]^{\Sigma_n}$$

which is:

$$F_p[\sum_i y_i^{p^{n-1}(p-1)}, \sum_{i,j} \prod_{i \neq j} (y_j y_i)^{p^{n-1}(p-1)}, \cdots, \prod_i (y_i)^{p^{n-1}(p-1)})]$$

Let Φ_n be the algebra map between P_n' and $P_n^{B_n}$ given by $\Phi_n\left(y_i^{p^{i-1}(p-1)}\right) = h_i^{p-1}$. This is an algebra isomorphism but not a Steenrod algebra map. Then $\Phi_n(S_n') = P_n^{GL_n}$, the so called Dickson algebra, abbreviated by D_n.

Let the generators for the Dickson algebra be $\{d_{n,0}, ..., d_{n,n-1}\}$. Then because of the isomorphism above, the following relations are deduced:

$$d_{n,n-i} = \sum_{1 \leq j_1 < \cdots < j_i \leq n} \prod_{s=1}^{i} \left(h_{j_s}^{p-1}\right)^{p^{n-i+s-j_s}} \tag{2.2}$$

Moreover, using 2.1 and 2.2 we deduce the following well known formula:

$$h_i = \sum_{t=0}^{i-1} (-1)^{i-1-t} y_i^{p^t} d_{i-1,t} \tag{2.3}$$

Here $d_{i-1,i-1} := 1$. Let us also note that the last two formulas will be of great importance in the sequel.

3. The Steenrod algebra action on Dickson invariants

The action mentioned above has been given for $p = 2$ in [1] and [3]. For p odd, it has been calculated only for the Steenrod algebra generators, P^{p^i}, and particular cases ([3]). Extending the idea used in the previous section, we compute it for any element P^k on the two extreme generators of D_n, namely $d_{n,n-1}$ and $d_{n,0}$.

Let us start with $P^k d_{n,n-1}$. We recall that

$$P^k y_i^n = \binom{n}{k} y^{n+k(p-1)} \tag{3.1}$$

In particular,

$$P^k y_i^{p^n} = \begin{cases} y_i^{p^n} & \text{if } k = 0 \\ y_i^{p^{n+1}} & \text{if } k = p^n \\ 0 & \text{otherwise} \end{cases}$$

Let us define an ordering between sequences $I = (i_n, ..., i_1)$ and $J = (j_n, ..., j_1)$ such that $I > J$ iff $i_t > j_t$ and t is the biggest index with this property; otherwise $I = J$. Next we consider sequences between exponents of various monomials in P_n. The biggest exponent in $d_{n,n-1} = \left(h_n^{p-1} + h_{n-1}^{p(p-1)} + ... + h_1^{p^{n-1}(p-1)}\right)$ is $(p^{n-1}(p-1), 0, ..., 0)$ which is associated with the monomial $y_n^{p^{n-1}(p-1)}$. Let us consider a typical monomial in D_n, $d_{n,0}^{a_0} \cdots d_{n,n-1}^{a_{n-1}}$, its biggest sequence is the following:

$$\left(p^{n-1}(p-1)\left(\sum_0^{n-1} a_t\right), p^{n-2}(p-1)\left(\sum_0^{n-2} a_t\right), \cdots, (p-1)a_0\right) \tag{3.2}$$

We are interested in those natural numbers k such that $P^k d_{n,n-1} = c d_{n,0}^{a_0} \cdots d_{n,n-1}^{a_{n-1}}$ for $c \neq 0$. For degree reasons: $k = p^{n-1}\left(\sum_0^{n-1} a_t - 1\right) + p^{n-2}\left(\sum_0^{n-2} a_t\right) + \cdots + a_0$ and $\sum_0^{n-1} a_t \leq p-1$. Let $k = \sum_1^n k_t$ and P^{k_t} applies to $y_t^{m_t}$. Let $A_s = a_0 + \cdots + a_s$ and b_t a non-negative integer such that $0 \leq b_t \leq \min[(p-1)b_{t+1}, A_{t-1}]$. Here $s = 0, \cdots, n-1$ and $t = 1, \cdots, n$.

EXAMPLE 1. *Let $n = 3$. We are looking for a k and a c such that $P^k d_{3,2} = P^k\left(h_3^{p-1} + h_2^{p(p-1)} + h_1^{p^2(p-1)}\right) = c d_{3,0}^{a_0} d_{3,1}^{a_1} d_{3,2}^{a_2}$. Let us concentrate only on*

$$P^k\left(h_3^{(p-1)}\right) = P^k\left(y_3^{p^2} - y_3^p d_{2,1} + y_3 d_{2,0}\right)^{(p-1)} \tag{3.3}$$

Of course, $k = k_1 + k_2 + k_3 = p^2(a_0 + a_1 + a_2 - d) + p(a_0 + a_1) + a_0$. Let a summand in 3.3 be $P^k\left((-1)^{b_{3,1}}\binom{(p-1)}{b_{3,1}+b_{3,0}}\binom{b_{3,1}+b_{3,0}}{b_{3,1}} y_3^{p^2((p-1)-b_{3,1}-b_{3,0})+pb_{3,1}+b_{3,0}} d_{2,1}^{b_{3,1}} d_{2,0}^{b_{3,0}}\right) =$ $(-1)^{b_{3,1}}\binom{(p-1)}{b_{3,1}+b_{3,0}}\binom{b_{3,1}+b_{3,0}}{b_{3,1}} P^{k_3} y_3^{p^2((p-1)-b_{3,1}-b_{3,0})+pb_{3,1}+b_{3,0}} P^{k_2} d_{2,1}^{b_{3,1}} P^{k_1} d_{2,0}^{b_{3,0}}$. Next, we consider such a k_3 such that
$P^{k_3} y_3^{p^2((p-1)-b_{3,1}-b_{3,0})+pb_{3,1}+b_{3,0}} = \binom{p^2((p-1)-b_{3,1}-b_{3,0})+pb_{3,1}+b_{3,0}}{p^2(a_0+a_1+a_2-1)+p(b_{3,1}+b_{3,0})+b_{3,0}} y_3^{(p-1)p^2(a_0+a_1+a_2)}$
This is identically non zero only if $b_{3,0} = 0$. Hence $k_3 = p^2(a_0 + a_1 + a_2 - 1) + pb_{3,1}$ and the summation runs over $0 \leq b_{3,1} \leq p - (a_0 + a_1 + a_2)$. But $k - k_3 = p(a_0 + a_1 - b_{3,1}) + a_0$ implies that $b_{3,1} \leq a_0 + a_1$. Thus $0 \leq b_{3,1} \leq \min[p - (a_0 + a_1 + a_2), a_0 + a_1]$. Next we consider $P^{k_2+k_1} d_{2,1}^{b_{3,1}} = P^{k_2+k_1}(h_2^{p-1} + h_1^{p(p-1)})^{b_{3,1}}$. The only eligible summand for our target is $P^{k_2+k_1}(h_2^{p-1})^{b_{3,1}}$ (otherwise, the exponent of y_1 exceeds the required one). $P^{k_2+k_1}(y_2^p - y_2 y_1^{(p-1)})^{(p-1)b_{3,1}}$. As before:
$P^{k_2} y_2^{p((p-1)b_{3,1}-b_2)+b_2} P^{k_1} y_1^{(p-1)b_2} = \binom{p((p-1)b_{3,1}-b_2)+b_2}{p(a_0+a_1-b_{3,1})+b_2}\binom{(p-1)b_2}{a_0-b_2} y_2^{(p-1)p(a_0+a_1)} y_1^{(p-1)a_0}$
Hence $k_2 = p(a_0 + a_1 - b_{3,1}) + b_2$ and $k_1 = a_0 - b_2$. And the summation runs over $0 \leq b_2 \leq \min((p-1)b_{3,1}, a_0)$. Finally the coefficient of the required element is given by the following sum:

$$\sum_{b_{3,1}, b_2} (-1)^{b_{3,1}+b_2}\binom{(p-1)}{b_{3,1}}\binom{(p-1)b_{3,1}}{b_2}\binom{(p-1)-b_{3,1}}{a_0+a_1+a_2-1}\binom{p((p-1)b_{3,1}-b_2)+b_2}{p(a_0+a_1-b_{3,1})+b_2}\binom{(p-1)b_2}{a_0-b_2}$$

THEOREM 1. *Let* $k = p^{n-1}\left(\sum_0^{n-1} a_t - 1\right) + p^{n-2}\left(\sum_0^{n-2} a_t\right) + \cdots + a_0$ *with* $\sum_0^{n-1} a_t \le p-1$. *Then* $P^k d_{n,n-1} = c d_{n,0}^{a_0} \cdots d_{n,n-1}^{a_{n-1}}$, *where c is the following constant* mod $-p$.
$\sum_{b_t}(-1)^b \binom{p-1}{b_n}\binom{(p-1)b_n}{b_{n-1}} \cdots \binom{(p-1)b_3}{b_2}\binom{p-1-b_n}{A_{n-1}-1}\binom{(p-1)b_n-b_{n-1}}{A_{n-2}-b_n} \cdots \binom{(p-1)b_t-b_{t-1}}{A_{t-2}-b_t} \cdots \binom{(p-1)b_2}{A_0-b_2}$

Here $b = \sum_1^{n-1} b_t$.

PROOF. It suffices to consider $P^k(h_n^{(p-1)})$. We prove the corresponding formula for $P^k(h_n^{(p-1)d})$ by induction on n. Here $k = p^{n-1}\left(\sum_0^{n-1} a_t - d\right) + p^{n-2}\left(\sum_0^{n-2} a_t\right) + \cdots + a_0$, $1 \le d \le p-1$ and $\sum_0^{n-1} a_t \le p-1$. The case $n = 3$ has been worked out in the last example. Let us recall that we are looking for the coefficient of the monomial $\prod_{i=1}^{n} y_i^{(p-1)p^{i-1}\left(\sum_0^{i-1} a_t\right)}$ after applying P^k. Expanding $(h_n)^{(p-1)d} = \left(\sum_{t=0}^{n-1}(-1)^{n-1-t} y_n^{p^t} d_{n-1,t}\right)^{(p-1)d}$ and considering the coefficient of

$$P^{k_n} y_n^{p^{n-1}[(p-1)d - b_{n,i_1} - \cdots - b_{n,i_{p-1}}] + p^{i_1-1} b_{n,i_1} + \cdots + p^{i_{p-1}-1} b_{n,i_{p-1}}} \tag{3.4}$$

We conclude that only $b_n := b_{n,n-1} \ne 0$. Moreover, $0 \le b_{n,n-1} \le (p-1)d$. Thus we proceed to the following element:

$$\sum_{b_n}(-1)^{b_n}\binom{(p-1)\,d}{b_n}\binom{(p-1)d - b_n}{A_{n-1}-1} y_n^{(p-1)p^{n-1}\left(\sum_0^{n-1} a_t\right)} P^{k-k_n} d_{n-1,n-2}^{b_n} \tag{3.5}$$

Here $k - k_n = p^{n-2}\left(\sum_0^{n-2} a_t - b_n\right) + p^{n-3}\left(\sum_0^{n-3} a_t\right) + \cdots + a_0$. ∎

Let us proceed to the case $P^k d_{n,0} = c \prod_{t=0}^{n-1} d_{n,t}^{a_t}$. Restrictions imply that $k = \sum_0^{n-1}(a_0 + \cdots + a_{t-1}) p^t \le p^n - 1$ and $a_0 \ge 1$. The biggest exponents in $d_{n,0}$ and $c \prod_{t=0}^{n-1} d_{n,t}^{a_t}$ are $(p^{n-1}(p-1), \cdots, (p-1))$ and $(p^{n-1}(p-1)(a_0 + \cdots + a_{n-1}), \cdots (p-1)a_0)$ respectively. The idea is to consider all monomials f in $d_{n,0} = \prod h_i^{p-1} = \left(\sum_\sigma [y]^{p^\sigma}\right)^{p-1}$ such that $P^k f = c_f [y]^{(p^{n-1}(p-1)(a_0+\cdots+a_{n-1}), \cdots (p-1)a_0)}$ and c_f non-identically zero. Here σ is a permutation on $\{0, \cdots, n-1\}$ and $[y]^{p^\sigma} = y_n^{p^{\sigma_0}} \cdots y_1^{p^{\sigma_{n-1}}}$. All coefficients c_f are added and that constant will be the coefficient c of $\prod_{t=0}^{n-1} d_{n,t}^{a_t}$ in $P^k d_{n,0}$. Note that this decomposition holds for this particular generator only. Let us recall that an element of the form $[y]^{p^\sigma}$ is called a Peterson polynomial and $\prod h_i$ contains all such polynomials of the given degree $1 + \cdots + p^{n-1}$. Among those, we consider only the ones with the right degree called **leading Peterson's polynomials**. Our first task is to find the leading Peterson's polynomials. The cardinality of this set of polynomials is given by the n-th element of a Fibonacci sequence.

PROPOSITION 2. *Let $[y]^{p^I} = y_n^{p^{i_n}} \cdots y_1^{p^{i_1}}$, for $0 \le i_t \le n-1$. If $t < i_t$ or $i_t < t-2$, then $P^k[y]^{p^I}$ does not contain a multiple of $[y]^{(p^{n-1}(p-1)(a_0+\cdots+a_{n-1}),\cdots(p-1)a_0)}$ where $k = \sum_0^{n-1} (a_0 + \cdots + a_{t-1})\, p^t$.*

PROOF. Let us recall that $d_{n,0} = \left(\sum_\sigma [y]^{p^\sigma}\right)^{p-1} = \sum c_{(I_1,\cdots,I_{p-1})} \prod_{s=1}^{p-1} [y]^{p^{I_s}}$. Here $I_s = (i_{n,s}, \cdots, i_{1,s})$. A Steenrod operation acts on monomials by Cartan formula: $k = k_n + \cdots + k_1$. Let us consider $P^{k_t} y_t^{\sum p^{i_{s,t}} e_s}$ a typical summand in $d_{n,0}$. Here $1 \le e_s \le p-1$.

(3.6) $\quad \sum p^{i_{s,t}} e_s + k_t(p-1) = p^{t-1}(p-1)(a_0 + \cdots + a_{t-1}) \Longrightarrow i_{s,t} \le t$

Let $\sum p^{i_{s,t}} e_s = p^t E_t + p^{t-1} E_{t-1} + \cdots + pE_1 + (p-1-E_t - \cdots - E_1)$, then

(3.7)

$k_t = p^{t-1}(a_0 + \cdots + a_{t-1} - E_t - 1) + p^{t-2}(p-1-E_t - E_{t-1}) + \cdots + (p-1-E_t - \cdots - E_1)$

Hence Steenrod's binomial coefficients are as follows:

(3.8) $$\binom{E_{t-1}}{a_0 + \cdots + a_{t-1} - E_t - 1}\binom{E_{t-2}}{p-1-E_t-E_{t-1}} \cdots \binom{E_1}{p-1-E_t - \cdots E_2}$$

The claimed restrictions are induced: $E_t + E_{t-1} + 1 \ge a_0 + \cdots + a_{t-1}$ and $a_0 + \cdots + a_{t-1} \ge E_t + 1$. $p-1 = E_t + E_{t-1} + E_{t-2} \Longrightarrow E_{t-s} = 0$ for $s > 2$. ∎

DEFINITION 1. *A Peterson polynomial $y_n^{p^{i_n}} \cdots y_1^{p^{i_1}}$ satisfying $n-2 \le i_n \le n-1$, $t-2 \le i_t \le t$ and $0 \le i_1 \le 1$ is called a leading Peterson polynomial. The set of leading Peterson polynomials is denoted by LPP_n.*

LEMMA 1. *The size of LPP_n is given by the n-th element of a Fibonacci sequence.*

PROOF. Let $n = 2$, then there are only two pairs of exponents: $(1,0)$ and $(0,1)$, $F_2 = 2$. Let $n = 3$, then there are only three triples of exponents: $(2,1,0)$, $(2,0,1)$ and $(1,2,0)$, $F_3 = 3$. Given F_k for $k < n$, F_n counts sequences of the form $(n-1, i_{n-1}, \cdots, i_1)$ plus $(n-2, n-1, i_{n-2}, \cdots, i_1)$. The size of the first set is F_{n-1} and the second F_{n-2}. ∎

Our problem reduces to the case $P^k(LPP_n)^{p-1}$.

THEOREM 2. *Let $k = \sum_0^{n-1} (a_0 + \cdots + a_{t-1})\, p^t$, then $P^k d_{n,0} = c \prod_{t=0}^{n-1} d_{n,t}^{a_t}$ where the coefficient c is given by:*

(3.9) $$\left[\sum_{\substack{I_t \in ELPP_n \\ E_1 + \cdots E_{p-1} = p-1}} \frac{(p-1)!}{E_1! \cdots E_{p-1}!} \prod_{t=1}^{n} \binom{B_{t-1,t}}{a_0 + \cdots + a_{t-1} - 1 - B_{t,t}}\right] \bmod p$$

Here $ELPP_n$ is the set which contains all exponents of monomials from LPP_n and $B_{t,s}$ is defined inductively as follows:

$B_{n-1,n} = \sum E_s(i_{n,s} - (n-2))$;

$B_{n-2,n} = p - 1 - B_{n-1,n};$
$B_{t-1,t-1} = p - 1 - B_{t-1,t} - B_{t-1,t+1};$
$B_{t-2,t-1} = \sum E_s(i_{t-1,s} - (t-3)) - B_{t-1,t-1};$
$B_{t-3,t-1} = p - 1 - B_{t-1,t-1} - B_{t-2,t-1}.$

PROOF. Since $k < p^n$, the value of P^k on $d_{n,0}$ is a monomial. Last proposition implies that all coefficients in $P^k(LPP_n)^{p-1}$ must be added up. It remains to define the terms $B_{t,s}$. Let us recall that we are considering monomials of the form $\prod_t (y_n^{p^{i_{n,t}}} \cdots y_1^{p^{i_{1,t}}})^{E_t}$. For each variable, the coefficients of powers of its exponents add up to $p-1$. Let us consider y_n: $B_{n-1,n}p^{n-1} + B_{n-2,n}p^{n-2} = \sum p^{i_{n,s}} E_s$. Then $B_{n-1,n} = \sum E_s(i_{n,s} - (n-2))$ and $B_{n-2,n} = p - 1 - B_{n-1,n}$. For y_{n-1}, we have the following equation $B_{n-1,n-1}p^{n-1} + B_{n-2,n-1}p^{n-2} + B_{n-3,n-1}p^{n-3} = \sum p^{i_{n-1s}} E_s$ which implies $B_{n-1,n-1} = p - 1 - B_{n-1,n}$, $B_{n-2,n-1} = \sum E_s(i_{n-1,s} - (n-3)) - B_{n-1,n-1}$ and $B_{n-3,n-1} = p - 1 - B_{n-2,n-1} - B_{n-1,n-1}$. Now the claimed formulas for $B_{t,s}$ are easily deduced. ∎

Let us recall the analogue formula for $P^k(h_n)^{p-1}$ from theorem 10 page 950 in [3].

THEOREM 3. [3]
(3.10)

$$P^k(h_n)^{p-1} = \begin{cases} \frac{1}{d_{n-1,0}} \left(P^k d_{n,0} - h_n{}^{p-1}(P^k d_{n-1,0}) + \sum_{m=0}^{n-2} d_{n-1,n-2-m} P^{k-p_m} d_{n-1,0} \right) \\ 0, \ if \ k \neq \sum_{m=0}^{n-1} c_m p_m \end{cases}$$

Here $p_m = p^{n-1} + \cdots + p^{n-1-m}$.

Using formula 2.2, Cartan formula, and the two last theorems, the interested reader can evaluate $P^k d_{n,s}$ for $0 < s < n-1$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOANNINA, IOANNINA 45110, GREECE
E-mail address: nkechag@cc.uoi.gr
URL: http://www.uoi.gr/schools/scmath/math/staff/nkechag/

Strict Topologies and Vector-Measures on non-Archimedean Spaces

A. K. KATSARAS Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece

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Introduction

Let $C_b(X,E)$ be the space of all bounded continuous functions from a zero-dimensional Hausdorff topological space X to a non-Archimedean Hausdorff locally convex space E. In section 2 of this paper, we look at some of the properties of the locally convex topologies β, β', β_1 and β_1' on $C_b(X,E)$, introduced by the author in [8], and we show that the corresponding dual spaces are certain subspaces of a space $M(X,E')$ of finitely-additive E'-valued measures on the algebra of all clopen subsets of X introduced in [6]. In case E is a polar space, it is proved that the strict topology β_o, which was defined by the author in [7], coincides with the polar topology associated with β'. In section 3 we look at the supports of members of $M(X,E')$ and in section 4 we introduce the topologies β_e and β_e'. In case E is metrizable, it is shown that β_e is coarser than β_1 and coincides with the topology of simple convergence on uniformly bounded equicontinuous subsets of $C_b(X,E)$. In section 5 we look at the dual spaces of $C_b(X,E)$ under the topologies β_u and β_u', which were defined in [1] and [3], respectively, while in section 6 we investigate the dual spaces for the topologies β_e and β_e'. When E is metrizable, it is proved that β_e' yields as dual space the space of the so called separable members of $M(X,E')$ and that the same does β_u'. Moreover the two topologies have the same equicontinuous sets in their common dual space.

1 Preliminaries

Throughout this paper, $\mathbb{K}$ stands for a complete non-Archimedean valued field whose valuation is non-trivial. By a seminorm, on a vector space E over $\mathbb{K}$, we mean a non-Archimedean seminorm. Similarly, by a locally convex space we mean a non-Archimedean locally convex space over $\mathbb{K}$. For E a locally convex space, we denote by $cs(E)$ the collection of all continuous seminorms on E and by E' its dual space.

Let now X be a zero-dimensional Hausdorff topological space and E a Hausdorff locally convex space. We will denote by $\beta_o X$ the Banaschewski compactification of X (see [4]) and by $\upsilon_o X$ the $\mathbb{N}$-repletion of X ($\mathbb{N}$ is the set of natural numbers), i.e. the subspace of $\beta_o X$ consisting of all $x \in \beta_o X$ with the following property: For each sequence (V_n) of neighborhoods of x in $\beta_o X$ we have that $\bigcap V_n \cap X \neq \emptyset$. The space X is called $\mathbb{N}$-replete if $X = \upsilon_o X$. We will denote by $C_b(X, E)$ the space of all bounded continuous E-valued functions on X and by $C_{rc}(X, E)$ the space of all $f \in C_b(X, E)$ for which $f(X)$ is relatively compact in E. In case $E = \mathbb{K}$, we will simply write $C_b(X)$ and $C_{rc}(X)$ respectively. For $A \subset X$, we denote by χ_A the $\mathbb{K}$-valued characteristic function of A in X and by $\overline{A}^{\beta_o X}$ the closure of A in $\beta_o X$. Every $f \in C_{rc}(X, E)$ has a unique continuous extension f^{β_o} to all of $\beta_o X$. For f an E-valued function on X, p a seminorm on E and $A \subset X$, we define

$$\|f\|_p = \sup_{x \in X} p(f(x)), \quad \|f\|_{A,p} = \sup_{x \in A} p(f(x)).$$

The strict topology β_o on $C_b(X, E)$ (see [7]) is the locally convex topology generated by the seminorms $f \mapsto \|hf\|_p$, where $p \in cs(E)$ and h is in the space $B_o(X)$ of all bounded $\mathbb{K}$-valued functions on X which vanish at infinity, i.e. for each $\epsilon > 0$ there exists a compact subset Y of X such that $|h(x)| < \epsilon$ if x is not in Y. As it is shown in [7], β_o has the same bounded sets with the topology τ_u of uniform convergence, i.e. the topology generated by the seminorms $\|.\|_p, p \in cs(E)$. Also β_o coincides with the topology τ_k of compact convergence on τ_u-bounded subsets of $C_b(X, E)$.

Let now $K(X)$ be the algebra of all clopen, (i.e. closed and open) subsets of X. We denote by $M(X, E')$ (see [6]) the space of all finitely-additive E'-valued measures m on $K(X)$ for which $m(K(X))$ is an equicontinuous subset of E'. For each m in $M(X, E')$ there exists $p \in cs(E)$ with $m_p(X) < \infty$, where, for $A \in K(X)$,

$$m_p(A) = \sup\{|m(B)s|/p(s) : p(s) \neq 0, A \supset B \in K(X)\}.$$

The space of all $m \in M(X, E')$ with $m_p(X) < \infty$ is denoted by $M_p(X, E')$. Next, we recall the definition of the integral of an E-valued function f on X with respect to an $m \in M(X, E')$. For $A \in K(X), A \neq \emptyset$, let $\mathcal{D}_A$ denote the family of all $\alpha = \{A_1, \ldots, A_n : x_1, \ldots, x_n\}$, where $\{A_1, \ldots, A_n\}$ is a clopen partition of A and $x_i \in A_i$. We make $\mathcal{D}_A$ a directed set by defining $\alpha_1 \geq \alpha_2$ iff the partition of A in α_1 is a refinement of the one in α_2. For $f \in E^X, m \in M(X, E')$ and $\alpha = \{A_1, \ldots, A_n : x_1, \ldots, x_n\}$, we define $\omega_\alpha(f, m) = \sum_{i=1}^n m(A_i) f(x_i)$. If the $\lim_\alpha \omega_\alpha(f, m)$ exists in $\mathbb{K}$, we will say that f is m-integrable over A and denote this limit by $\int_A f dm$. We define the integral over the empty set to be 0. For $A = X$, we write simply $\int f dm$. It is easy to see that if f is m-integrable over X, then it is m-integrable over every $A \in K(X)$ and $\int_A f dm = \int \chi_A f dm$. Every $m \in M(X, E')$ defines a τ_u-continuous linear functional on $C_{rc}(X, E)$ by $f \mapsto \int f dm$ (see [6]). Also every $\phi \in (C_{rc}(X, E), \tau_u)'$ is given in this way by a unique m.

For $p \in cs(E)$, we denote by $M_{t,p}(X, E')$ the space of all $m \in M_p(X, E')$ for which m_p is tight, i.e. for every $\epsilon > 0$, there exists a compact subset Y of X such that $m_p(A) \leq \epsilon$ if A is disjoint from Y. We define

$$M_t(X, E') = \bigcup_{p \in cs(E)} M_{t,p}(X, E').$$

As it is shown in [7], every $m \in M_t(X, E')$ defines a β_o-continuous linear form on $C_b(X, E)$ by $u_m(f) = \int f dm$. Moreover the map $m \mapsto u_m$, from $M_t(X, E')$ to $(C_b(X, E), \beta_o)'$, is an algebraic isomorphism. Finally we recall that a locally convex space E has the countable neighborhood property if, for each sequence (p_n) of continuous seminorms on E, there exist a $p \in cs(E)$ and a sequence (α_n) of positive numbers such that $p \geq \alpha_n p_n$ for all n. For all unexplained terms on locally convex spaces, we refer to [15] and [16].

Throughout the paper, X is a zero-dimensional Hausdorff topological space and E a Hausdorff locally convex space.

2 On the Topologies $\beta, \beta', \beta_1, \beta_1'$

We recall the definitions of the locally convex topologies β, β', β_1 and β_1' on $C_b(X,E)$ introduced by the author in [8]. Let $\Omega = \Omega(X)$ be the family of all compact subsets of $\beta_o X$ which are disjoint from X. For $H \in \Omega$, let C_H be the space of all $h \in C_{rc}(X)$ whose the continuous extension h^{β_o} vanishes on H. For $p \in cs(E)$, let $\beta_{H,p}$ denote the locally convex topology on $C_b(X,E)$ generated by the seminorms $\|.\|_{h,p}$, where $h \in C_H$ and $\|f\|_{h,p} = \|hf\|_p$. The inductive limit of the topologies $\beta_{H,p}$, as H ranges over Ω, is denoted by β_p, while β' is the projective limit of the topologies β_p, $p \in cs(E)$. Also, for $H \in \Omega$, β_H is the locally convex topology generated by the seminorms $\|.\|_{h,p}, h \in C_H, p \in cs(E)$. The inductive limit of the topologies $\beta_H, H \in \Omega$, is denoted by β. Replacing Ω by the family Ω_1 of all $\mathbb{K}$-zero subsets of $\beta_o X$ which are disjoint from X, we get the topologies $\beta_{1,p}$, for $p \in cs(E), \beta_1$ and β_1'. Recall that a $\mathbb{K}$-zero subset of $\beta_o X$ is a set of the form $\{x \in \beta_o X : g(x) = 0\}$ for some $g \in C(\beta_o X)$. Analogous with topologies β and β' are the topologies β_u and β_u' which were defined in [3]. They are obtained by replacing Ω by the family Ω_u of all $Q \in \Omega$ with the following property: There exists a clopen partition $(A_i)_{i \in I}$ of X with $\overline{A_i}^{\beta_o X}$ disjoint from Q for all $i \in I$.

Theorem 2.1 [8]. *For $H \in \Omega$ and $p \in cs(E)$, $\beta_{H,p}$ has as a base at zero the sets of the form*

$$\bigcap_{n=1}^{\infty} \{f \in C_b(X,E) : \|f\|_{A_n,p} \leq \alpha_n\},$$

where (α_n) is an increasing sequence of positive numbers, tending to ∞, and (A_n) an increasing sequence of clopen subsets of X with $\overline{A_n}^{\beta_o X}$ disjoint from H for all n.

We only sketch the proof of the next Theorem since it is a modification of the proof of Theorem 4.1 in [8].

Theorem 2.2 *An absolutely convex subset V of $C_b(X,E)$ is a $\beta_{H,p}$-neighborhood of zero iff the following condition is satisfied: For each $r > 0$, there exists a clopen subset A of X, with $\overline{A}^{\beta_o X}$ disjoint from H, and $\epsilon > 0$ such that*

$$\{f \in C_b(X,E) : \|f\|_p \leq r, \|f\|_{A,p} \leq \epsilon\} \subset V.$$

Proof: The necessity follows using the preceding Theorem. Conversely, suppose that the condition is satisfied and let $\lambda \in \mathbb{K}, |\lambda| > 1$. Choose an increasing sequence (A_n) of clopen sets, with $\overline{A_n}^{\beta_o X}$ disjoint from H, and a decreasing sequence (ϵ_n) of positive numbers, $\epsilon_n \to 0$, such that $U_n \cap \lambda^n U \subset V$, where

$$U_n = \{f \in C_b(X,E) : \|f\|_{A_n,p} \leq \epsilon_n\}, \quad U = \{f \in C_b(X,E) : \|f\|_p \leq 1\}.$$

Let $V_1 = U_1 \bigcap [\bigcap_{n=1}^{\infty} (U_{n+1} + \lambda^n U)]$. Then $V_1 \subset V$. Choose $\lambda_1 \in \mathbb{K}$, with $0 < \lambda_1| < \min\{1, \epsilon_1\}$, and take $\lambda_n = \lambda^{n-1}$ for $n > 1$. Now

$$\bigcap_{n=1}^{\infty} \{f \in C_b(X, E) : \|f\|_{A_n,p} \leq |\lambda_n|\} \subset V_1,$$

and hence the result follows from the preceding Theorem.

Corollary 2.3 *If $\tau_{u,p}$ is the topology generated by the seminorm $\|.\|_p$, then β_p is the finest loally convex topology on $C_b(X, E)$ which coincides with β_p on $\tau_{u,p}$-bounded sets.*

We will show next that the dual space of $C_b(X, E)$, under the topology β, is a certain subspace of $M(X, E')$. Let $M_\tau(X, E')$ be the space of all $m \in M(X, E')$ with the following property: For each net (A_δ) of clopen subsets of X which decreases to the empty set, there exists $p \in cs(E)$, with $m_p(X) < \infty$, such that $m_p(A_\delta) \to 0$. Replacing decreasing nets by decreasing sequences, we get the space $M_\sigma(X, E')$

Theorem 2.4 *If $m \in M_\tau(X, E')$, then every member of $C_b(X, E)$ is m-integrable and the linear map $u_m : C_b(X, E) \to \mathbb{K}, u_m(f) = \int f dm$, is β-continuous.*

Proof: There exists $p \in cs(E)$ with $m_p(X) \leq 1$. Let $f \in C_b(X, E)$ and $\epsilon > 0$. We may assume that $\|f\|_p \leq 1$. Let $(A_i)_{i \in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $p(f(x) - f(y)) \leq \epsilon$. Choose $x_i \in A_i$. The function $f^* = \sum_i \chi_{A_i} f(x_i)$ is continuous. For each finite subset J of I, set $B_J = \bigcup_{i \notin J} A_i$. Then B_J is clopen and $B_J \downarrow \emptyset$. By our hypothesis, there exists $q \in cs(E), q \geq p$, such that $m_q(B_J) \to 0$. Choose a finite subset J of I such that $m_q(B_J) < \epsilon / \|f\|_q$. Let $g = \sum_{i \in J} \chi_{A_i} f(x_i)$ and $h = f^* - g$. For any clopen partitioin $\{D_1, \ldots, D_n\}$ of X, which is a refinement of $\{A_i \in J\} \cup \{B_J\}$, and any $y_k \in D_k$, we have

$$\left| \sum_{k=1}^{n} m(D_k) h(y_k) \right| \leq \epsilon \text{ and } \sum_{k=1}^{n} m(D_k) g(y_k) = \sum_{i \in J} m(A_i) f(x_i).$$

Thus

$$\left| \sum_{k=1}^{n} m(D_k) f^*(y_k) - \sum_{i \in J} m(A_i) f(x_i) \right| \leq \epsilon \text{ and } \left| \sum_{k=1}^{n} m(D_k) [f(y_k) - f^*(y_k)] \right| \leq \epsilon,$$

and so $\left| \sum_{k=1}^{n} m(D_k) f(y_k) - \sum_{i \in J} m(A_i) f(x_i) \right| \leq \epsilon$. It follows that f is m-integrable. Finally, u_m is β-continuous. Indeed, let $H \in \Omega$. It suffices to show that u_m is $\beta_{H,p}$-continuous for some $p \in cs(E)$. To this end, we first observe that there exists a decreasing net (B_δ) of clopen subsets of X with $\bigcap \bar{B_\delta}^{\beta_o X} = H$. Since $m \in M_\tau(X, E')$, there exists $p \in cs(E)$ such that $m_p(X) \leq 1$ and $\lim m_p(B_\delta) = 0$. We will show that u_m is $\beta_{H,p}$-continuous. Let $W = \{f \in C_b(X, E) : |u_m(f)| \leq 1\}$ and $r > 0$. There exists δ with $m_p(B_\delta) < 1/r$. If $B = X \setminus B_\delta$, then $\bar{B}^{\beta_o X}$ is disjoint from H and

$$\{f \in C_b(X, E) : \|f\|_p \leq r, \|f\|_{B,p} \leq 1\} \subset W.$$

The result now follows from Theorem 2.2.

Theorem 2.5 *The map $u : M_\tau(X, E') \to (C_b(X, E), \beta)', m \mapsto u_m$, is an algebraic isomorphism.*

Proof: It remains only to show that u is onto. So, let ϕ a β-continuous linear functional on $C_b(X, E)$. Since β is coarser that τ_u, there exists $m \in M(X, E')$ such that $\phi(f) = \int f dm$ for all $f \in C_{rc}(X, E)$. We will show that $m \in M_\tau(X, E')$. In fact, let (A_δ) be a net of clopen sets, which decreases to the empty set, and let $H = \bigcap \bar{A_\delta}^{\beta_o X}$. Since ϕ is β_H-continuous, there exist $p \in cs(E)$ and $h \in C_H$ such that

$$W_1 = \{f \in C_b(X, E) : \|hf\|_p \le 1\} \subset \{f : |\phi(f)| \le 1\}.$$

We will show that $m_p(A_\delta) \to 0$. So, let μ be a non-zero element of $\mathbb{K}$. The set

$$G = \{x \in \beta_o X : |h^{\beta_o}(x)| \le |\mu|\}$$

is clopen and contains H. There exists δ with $\bar{A_\delta}^{\beta_o X} \subset G$. If now A is a clopen subset of A_δ and $s \in E$ with $p(s) \le 1$, then $\mu^{-1}\chi_A s \in W_1$ and so $|m(A)s| \le |\mu|$. If $\lambda \in \mathbb{K}$ with $|\lambda| \le 1$, then $m_p(A_\delta) \le |\lambda\mu|$, which clearly proves that $m \in M_\tau(X, E')$. Finally, $\phi = u_m$ since both ϕ and u_m are β-continuous and coincide on the β-dense subset $C_{rc}(X, E)$ of $C_b(X, E)$.

Using arguments analogous to the ones used in the proofs of Theorems 2.4 asnd 2.5, we get the following

Theorem 2.6 *A subset H, of the dual space $M_\tau(X, E')$ of $(C_b(X, E), \beta)$, is β-equicontinuous iff the following condition is satisfied: For each net (A_δ) of clopen subsets of X, which decreases to the empty set, there exists $p \in cs(E)$ such that $\sup_{p \in H} m_p(X) < \infty$ and $\sup_{m \in H} m_p(A_\delta) \to 0$.*

Next we will look at the dual space of $(C_b(X, E), \beta')$. For $p \in cs(E)$, let $\mathcal{M}_{\tau,p}(X, E')$ be the space of all $m \in M_p(X, E')$ such that $m_p(A_\delta) \to 0$ for each net (A_δ) of clopen subsets of X which decreases to the empty set. Let

$$\mathcal{M}_\tau(X, E') = \bigcup_{p \in cs(E} \mathcal{M}_{\tau,p}(X, E').$$

Replacing nets by decreasing sequences of clopens sets, we get the spaces $\mathcal{M}_{\sigma,p}(X, E')$ and $\mathcal{M}_\sigma(X, E')$.

As in the proofs of Theorems 2.4 and 2.5, we get that, for $m \in \mathcal{M}_{\tau,p}(X, E')$, u_m is β_p-continuous and every $\phi \in (C_b(X, E), \beta_p)'$ is of the form u_m for some $m \in \mathcal{M}_{\tau,p}(X, E')$. Thus, we have the following

Theorem 2.7 *a) For each $p \in cs(E)$, the map*

$$T_p : \mathcal{M}_{\tau,p}(X, E') \to (C_b(X, E), \beta_p)', m \mapsto u_m,$$

is an algebraic isomorphism.
b) $\mathcal{M}_\tau(X, E')$ is algebraically isomorphic to the dual space of $(C_b(X, E), \beta')$.
c) A subset H of $\mathcal{M}_\tau(X, E')$ is β_p-equicontinuous iff $\sup_{p \in H} m_p(X) < \infty$ and $\sup_{m \in H} m_p(A_\delta) \to 0$ for each net (A_δ) of clopen subsets of X which decreases to the empty set.

We will show now that $\mathcal{M}_{\tau,p}(X,E') = M_{t,p}(X,E')$. We need the following

Lemma 2.8 *Let $\mathcal{F}$ be a family of clopen subsets of X and let $m \in \mathcal{M}_{\tau,p}(X,E')$. If W is a clopen subset of $\bigcup\{A : A \in \mathcal{F}\}$, then there exists $A \in \mathcal{F}$ such that $m_p(W) \le m_p(A)$.*

Proof: Since, for clopen sets A, B, we have $m_p(A \cup B) = \max\{m_p(A), m_p(B)\}$, we may assume that $\mathcal{F}$ is closed under finite unions. The family $\mathcal{B} = \{W \setminus A : A \in \mathcal{F}\}$ is downwards directed to the empty set. If $m_p(W) > 0$, there exists $A \in \mathcal{F}$ with $m_p(W \setminus A) < m_p(W)$ and so $m_p(W) = \max\{m_p(W \cap A), m_p(W \setminus A)\} = m_p(W \cap A) \le m_p(A)$, and the result follows.

The proof of the following Theorem is analogous to the proof of Theorem 7.6 in [16].

Theorem 2.9 *Let $m \in \mathcal{M}_{\tau,p}(X,E')$ and let*

$$N_{m,p} : X \to \mathbb{R}, N_{m,p}(x) = \inf\{m_p(A) : x \in A \in K(X)\}.$$

Then: a) $N_{m,p}$ is upper semicontinuous.
b) For each $\epsilon > 0$, the set $X_{m,p,\epsilon} = \{x \in X : N_{m,p}(x) \ge \epsilon\}$ is compact.

Proof: a) It suffices to show that, for each $\theta > 0$, the set $W = \{x \in X : N_{m,p}(x) < \theta\}$ is open. So let $x \in W$. There exists a clopen neighborhood V of x such that $m_p(V) < \theta$. Then $V \subset W$.
b) Let $\mathcal{F}$ be a clopen cover of $X_{m,p,\epsilon}$. Without loss of generality, we may assume that $\mathcal{F}$ is closed under finite unions. The set $M = X \setminus X_{m,p,\epsilon}$ is a union of clopen sets. Thus the family $\mathcal{V} = \{X \setminus (V \cup W) : V \in \mathcal{F}, W \in K(X), W \subset M\}$ is downwards directed to the empty set. Thus, there exists $V \in \mathcal{F}, W \in K(X), W \subset M$ such that $m_p(X \setminus (V \cup W)) < \epsilon$. Hence $X_{m,p,\epsilon} \subset V \cup W$ and so $X_{m,p,\epsilon} \subset V$, which completes the proof.

Theorem 2.10 $\mathcal{M}_{\tau,p}(X,E') = M_{t,p}(X,E')$.

Proof: Let $m \in \mathcal{M}_{\tau,p}(X,E'), \epsilon > 0$ and A a clopen set disjoint from the compact set $Y = X_{m,p,\epsilon}$. Every $x \in A$ has a clopen neighborhood V_x with $m_p(V_x) < \epsilon$. In view of Lemma 2.8, we have that $m_p(A) \le \epsilon$, which proves that m is tight.

Conversely, assume that m is tight and let (A_δ) be a net of clopen sets decreasing to the empty set. Given $\epsilon > 0$, let Y be a compact subset of X such that $m_p(A) \le \epsilon$ if the clopen set A is disjoint from Y. There exists some A_δ which is disjoint from Y and so $m_p(A_\delta) \le \epsilon$. Hence the result follows.

Corollary 2.11 $\mathcal{M}_\tau(X,E') = M_t(X,E')$.

Recall that a subset H of $M(X,E')$ is called tight (see [7], Definition 3.5) if there exists $p \in cs(E)$ such that: (1) $\sup_{m\in H} m_p(X) < \infty$.
(2) For every $\epsilon > 0$, there exists a compact subset Y of X such that $m_p(A) \le \epsilon$ for every $m \in H$ and every clopen set A disjoint from Y.
By Theorem 3.6 in [7], a subset H of $M_t(X,E')$ is tight iff it is β_o-equicontinuous.

Theorem 2.12 *A subset H of $\mathcal{M}_\tau(X,E')$ is β'-equicontinuous iff it is β_o-equicontinuous.*

Proof: Since β_o is coarser than β', it suffices to show that every β'-equicontinuous subset of $\mathcal{M}_\tau(X, E')$ is β_o-equicontinuous. So let H be such a set. Then H is β_p-equicontinuous for some $p \in cs(E)$. In view of Theorem 2.7, we have that $\sup_{m\in H} m_p(X) < \infty$. Define

$$N_{H,p} : X \to \mathbb{R}, N_{H,p}(x) = \inf\{\sup_{m\in H} m_p(V) : x \in V \in K(X)\}.$$

Using Theorem 2.7, we get (as in the proof of Theorem 2.9) that $N_{H,p}$ is upper-semicontinuous and the set $Y_{H,\epsilon} = \{x \in X : N_{H,p}(x) \geq \epsilon\}$ is compact for every $\epsilon > 0$. For each $V \in K(X)$ disjoint from $Y_{H,\epsilon}$ and each $m \in H$, we have that $m_p(V) \leq \epsilon$. This proves that H is tight and so it is β_o-equicontinuous. Hence the result follows.

Corollary 2.13 *If E is a polar space, then β_o is the polar topology associated with β'.*

Proof: When E is polar, the space $(C_b(X, E), \beta_o)$ is polar. Now the result follows from the preceding Theorem.

Next we will look at the dual space of $C_b(X, E)$ under the topologies β_1 and β_1' . We only sketch the proof of the following Theorem since it is analogous to the corresponding proof given in [16], p. 49, for the case $E = \mathbb{K}$.

Theorem 2.14 *If X is $\mathbb{N}$-replete and E metrizable, then $f(X)$ has non-measurable cardinal for every $f \in C_b(X, E)$.*

Proof: Let d be an ultrametric on E generating its topology. For each positive integer n, consider the equivalence relation $\sim=\sim_n$ on X defined by $x \sim y$ iff $d(f(x), f(y)) \leq 1/n$. Let B_n be a subset of X having only one point in common with each equivalence class. Since X is $\mathbb{N}$-replete, B_n has non-measurable cardinal (see [16], Theorem 2.10). Let $A_n = f(B_n)$. For each $z \in f(X)$ choose a $\bar{z} \in G = \prod A_n$ such that $\bar{z}_n \in A_n$ and $d(z, \bar{z}_n) \leq 1/n$ for each n. In this way we get a map $\phi : f(X) \to G, \phi(z) = \bar{z}$. Since each A_n has non-measurable cardinal, it follows that G has non-measurable cardinal. The result now follows from the fact that ϕ is one-to-one.

Using an argument analogous to the one used in [16], Theorem 7.1, we get the following

Theorem 2.15 *If X is $\mathbb{N}$-replete, then $\mathcal{M}_{\sigma,p}(X, E') = \mathcal{M}_{\tau,p}(X, E')$ for all $p \in cs(E)$.*

Theorem 2.16 *Assume that E is metrizable and let $m \in \mathcal{M}_\sigma(X, E')$. If $f \in C_b(X, E)$ is such that $f(X)$ has non-measurable cardinal, then f is m-integrable.*

Proof: The space $f(X)$ is $\mathbb{N}$-replete since it is ultraparacompact and has non-measurable cardinal (see [16], Theorem 2.18). Hence, there exists a continuous extension f^{v_o} of f to all of v_oX. Let $m^{v_o} : K(v_oX) \to E', m^{v_o}(A) = m(A \cap X)$. Then $m^{v_o} \in \mathcal{M}_{\sigma,p}(v_oX, E')$. Since v_oX is $\mathbb{N}$-replete, we have that $m^{v_o} \in \mathcal{M}_\tau(v_oX, E')$ (in view of the preceding Theorem). Thus f^{v_o} is m^{v_o}-integrable, from which it follows easily that f is m-integrable.

Theorem 2.17 *For every β_1'-continuous linear functional ϕ on $C_b(X, E)$ there exists $m \in \mathcal{M}_\sigma(X, E')$ such that $\phi(f) = \int f dm$ for all $f \in C_{rc}(X, E)$.*

Proof: Since β_1' is coarser than τ_u, there exists (by [6], Theorem 2.8) an $m \in M(X,E')$ such that $\phi(f) = \int f dm$ for all $f \in C_{rc}(X,E)$. Also, there exists $p \in cs(E)$ such that the set $\{f : |\phi(f)| \le 1\}$ is a $\beta_{1,p}$-neighborhood of zero in $C_b(X,E)$. Let now (A_n) be a sequence of clopen subsets of X, with $A_n \downarrow \emptyset$, and $Q = \bigcap \bar{A_n}^{\beta_o X}$. Let $\lambda \in \mathbb{K}$, with $|\lambda| > 1$, and let μ be a non-zero element of $\mathbb{K}$. There exist a clopen subset B of X, with $\bar{B}^{\beta_o X} \bigcap Q = \emptyset$, and $\epsilon > 0$ such that

$$\{f \in C_b(X,E) : \|f\|_p \le 1, \|f\|_{B,p} \le \epsilon\} \subset \{f : |\phi(f)| \le 1\}.$$

Let n be such that $\bar{B}^{\beta_o X} \bigcap \bar{A_n}^{\beta_o X} = \emptyset$. It follows now easily that $m_p(A_n) \le |\lambda\mu|$ and hence $m \in \mathcal{M}_{\sigma,p}(X,E')$. Thus the result follows.

Theorem 2.18 *Let E be metrizable and assume that $f(X)$ has non-measurable cardinal for every $f \in C_b(X,E)$. If $m \in \mathcal{M}_\sigma(X,E')$, then the linear functional u_m on $C_b(X,E)$ is β_1'-continuous.*

Proof: Let $m \in \mathcal{M}_{\sigma,p}(X,E')$. Under the hypotheses of the Theorem, every $f \in C_b(X,E)$ is m-integrable. Let $Q \in \Omega_1$. The exists a decreasing sequence (A_n) of clopen subsets of X with $\bigcap \bar{A_n}^{\beta_o X} = Q$. Let $r > 0$ and choose n such that $m_p(A_n) < 1/r$. If B is the complement of A_n in X, then $\bar{B}^{\beta_o X} \cap Q = \emptyset$ and

$$\{f \in C_b(X,E) : \|f\|_p \le r, \|f\|_{B,p} \le 1/m_p(X)\} \subset \{f : |u_m(f)| \le 1\} = W.$$

Thus W is a $\beta_{Q,p}$-neighborhood for every $Q \in \Omega_1$ and so u_m is $\beta_{1,p}$-continuous. This completes the proof.

Theorem 2.19 *If $\phi \in (C_b(X,E), \beta_1)'$, then there exists a unique $m \in M_\sigma(X,E')$ such that $\phi(f) = \int f dm$ for all $f \in C_{rc}(X,E)$.*

Proof: Since β_1 is coarser than τ_u, there exists a unique $m \in M(X,E')$ such that $\phi(f) = \int f dm$ for all $f \in C_{rc}(X,E)$. We neeed to show that $m \in M_\sigma(X,E')$. So, let (A_n) be a sequence of clopen sets which decreases to the empty set. Then $Q = \cap \bar{A_n}^{\beta_o X}$ is in Ω_1. Since ϕ is β_1-continuous, it is $\beta_{Q,p}$-continuous for some $p \in cs(E)$. Let $h \in C_Q$ be such that

$$W_1 = \{f \in C_b(X,E) : \|hf\|_p \le 1\} \subset \{f : |\phi(f)| \le 1\}.$$

Taking a $q \ge p$ if necessary, we may assume that $m_p(X) \le 1$. We will finish the proof by showing that $m_p(A_n) \to 0$. So, let $\lambda \in \mathbb{K}$ with $|\lambda| > 1$ and let μ be a non-zero element of $\mathbb{K}$. There exists n_o such that

$$\overline{A_{n_o}}^{\beta_o X} \subset \{x \in \beta_o X : |h^{\beta_o}(x)| \le |\mu\lambda^{-1}|\}.$$

It follows easily from this that $m_p(A_{n_o}) \le |\mu|$ and the result follows.

Theorem 2.20 *Let $m \in M(X,E')$ be such that every $f \in C_b(X,E)$ is m-integrable. Then, u_m is β_1-continuous iff $m \in M_\sigma(X,E')$.*

Poof: The necessity follows from the preceding Theorem. Conversely, assume that $m \in M_\sigma(X,E')$ and let $Q \in \Omega_1$. There exists a decreasing sequence (A_n) of clopen subsets of X

with $Q = \bigcap \overline{A_n}^{\beta_o X}$. Let $p \in cs(E)$ be such that $m_p(X) < \infty$ and $m_p(A_n) \to 0$. Given $r > 0$, choose n such that $m_p(A_n) < 1/r$. If $B = X \setminus A_n$, then $\bar{B}^{\beta_o X}$ is disjoint from Q and

$$\{f \in C_b(X,E) : \|f\|_p \leq r, \|f\|_{B,p} \leq 1/m_p(X)\} \subset \{f : |u_m(f)| \leq 1\}.$$

Thus the set $W = \{f \in C_b(X,E) : |u_m(f)| \leq 1\}$ is a β_Q neighborhood of zero, for each $Q \in \Omega_1$, and so u_m is β_1-continuous, which was to be proved.

If X, Y are Hausdorff zero-dimensional topological spaces, then every continuous function $h : X \to Y$ induces a linear map $T_h : C_b(Y,E) \to C_b(X,E), f \mapsto f \circ h$.

Theorem 2.21 *If $h : X \to Y$ is continuous, then the induced map T_h is $\beta - \beta, \beta_1 - \beta_1, \beta' - \beta'$ and $\beta_1' - \beta_1'$ continuous. In case E is polar, T_h is $\beta_o - \beta_o$ continuous.*

Proof: Let W be a convex β-neighborhood of zero in $C_b(X,E)$ and let $V = T_h^{-1}(W)$. Let $h^{\beta_o} : \beta_o X \to \beta_o Y$ be the continuous extension of h. Given $Q \in \Omega(Y)$, there exists a decreasing net (W_δ) of clopen subsets of $\beta_o Y$ with $\cap W_\delta = Q$. Let $V_\delta = (h^{\beta_o})^{-1}(W_\delta), H = \cap V_\delta$. Then $H \in \Omega(X)$. Since W is a β-neighborhood of zero, it is a $\beta_{H,p}$-neighborhood of zero for some $p \in cs(E)$. Thus, given $r > 0$, there exist a clopen subset A of X, whose closure in $\beta_o X$ is disjoint from H, and $\epsilon > 0$ such that

$$\{f \in C_b(X,E) : \|f\|_p \leq r, \|f\|_{A,p} \leq \epsilon\} \subset W.$$

There exists a δ such that $\bar{A}^{\beta_o X}$ is disjoint from V_δ. The set $B = Y \setminus W_\delta \cap Y$ is clopen in Y with $\bar{B}^{\beta_o Y} \cap Q = \emptyset$. Moreover

$$\{g \in C_b(Y,E) : \|g\|_p \leq r, \|g\|_{B,p} \leq \epsilon\} \subset V.$$

This (by Theorem 2.2) implies that V is a β_Q-neighborhood of zero, which proves that T_h is $\beta - \beta$ continuous. Since an absolutely convex subset of $C_b(X,E)$ is a β'-neighborhood of zero iff it is a β_p-neighgborhood for some $p \in cs(E)$, the proof of the $\beta' - \beta'$ continuity of T_h is analogous. Also the proofs for the cases of the topologies β_1 and β_1' are analogous since a subset of $\beta_o Y$ is a $\mathbb{K}$-zero set iff it is the intersection of a decreasing sequence of clopen subsets of $\beta_o Y$. Finally, if E is polar, then β_o is the polar topology associated with β'. Now the $\beta_o - \beta_o$ continuity of T_h follows from the fact that , if T is a continuous linear map between two locally convex spaces G_1, G_2, then T is also continuous with respect to the corresponding polar topologies on G_1, G_2.

3 Supports of Members of $M(X, E')$

Recall that a subset Y of X is a support set for an $m \in M(X,E')$ if $m(A) = 0$ for each clopen A disjoint from Y. Clearly Y is a support set for m iff $\bar{Y}$ is such a set. For an $m \in M(X,E')$, we define

$$supp(m) = \bigcap \{V \in K(X) : m(A) = 0 \ \text{ if } \ A \in K(X), A \cap V = \emptyset\}.$$

If $m \in M_\tau(X,E')$, then for each $s \in E$ the set function $ms : K(X) \to \mathbb{K}, (ms)(A) = m(A)s$ is in $M_\tau(X)$ and hence $supp(m)$ is a suppoprt set for m by Theorem 3.5 in [6].

Let now $m \in M_p(X,E')$. For B a subset of X, we define $m_{*,p}(B) = \sup inf_n m_p(V_n)$, where the supremum is taken over all decreasing sequences (V_n) of clopen sets with $\cap V_n \subset B$.

Theorem 3.1 *Let X be ultraparacompact and $m \in M(X, E')$. Then $m \in \mathcal{M}_{\tau,p}(X, E')$ iff $m_p(X) < \infty$, $supp(m)$ is Lindelöff and $m_{*,p}(X \setminus supp(m)) = 0$.*

Proof: Assume that $m \in \mathcal{M}_{\tau,p}(X, E')$ and let (A_n) be a decreasing sequence of clopen sets with $\cap A_n$ disjoint from $supp(m)$. The family $\mathcal{U} = \{V \in K(X) : m_p(X \setminus V) = 0\}$ is downwards directed and $\bigcap_{V \in \mathcal{U}} V = supp(m)$. Thus the family $\{A_n \cap V : n \in \mathbb{N}, V \in \mathcal{U}\}$ is downwards directed to the empty set. Given $\epsilon > 0$, there exist n and $V \in \mathcal{U}$ such that $m_p(A_n) = m_p(A_n \cap V) < \epsilon$, which proves that $m_{*,p}(X \setminus supp(m)) = 0$. Next, let $\mathcal{F}$ be a clopen cover of $supp(m)$. Since X is ultraparacompact, there exists a clopen partition $(A_i)_{i \in I}$ of X which is a refinement of the open cover $\mathcal{F} \bigcup \{X \setminus supp(m)\}$ of X. Let $I_1 = \{i \in I : A_i \cap supp(m) = \emptyset\}$ and $I_2 = I \setminus I_1$. Then $supp(m) \subset \bigcup_{i \in I_2} A_i$. For each finite subset J of I, let $D_J = \bigcup_{i \notin J} A_i$. Then $D_J \downarrow \emptyset$ and so $m_p(D_{J_o}) < \epsilon$ for some finite subset J_o of I. Clearly $m_p(A_i) < \epsilon$ if $i \notin J_o$. Thus the set $M = \{i \in I : m_p(A_i) \neq 0\}$ is countable, $supp(m) \subset \bigcup_{i \in M} A_i$ and $M \subset I_2$. Since each A_i, for $i \in I_2$, is contained in some member of $\mathcal{F}$, it is clear that $supp(m)$ is covered by a countable subfamily of $\mathcal{F}$ and so $supp(m)$ is Lindelöff.

Conversely, assume that $m_p(X) < \infty$, $supp(m)$ is Lindelöff and $m_{*,p}(X \setminus supp(m)) = 0$. Let (Z_α) be a net of clopen subsets of X decreasing to the empty set. There exists an increasing sequence (α_n) such that $supp(m) \subset \bigcup_{n=1}^{\infty} X \setminus Z_{\alpha_n}$. By our hypothesis, given $\epsilon > 0$, there exists n such that $m_p(Z_{\alpha_n}) < \epsilon$, which clearly completes the proof.

Theorem 3.2 *Let X be ultraparacompact and let $m \in \mathcal{M}_{\sigma,p}(X, E')$. Then $m \in \mathcal{M}_{\tau,p}(X, E')$ iff $supp(m)$ is Lindelöff and $m(A) = 0$ if the clopen set A is disjoint from $supp(m)$.*

Proof: Assume that the condition is satisfied and let (Z_n) be a decreasing sequence of clopen sets such that the set $Z = \cap V_n$ does not meet $supp(m)$. Since X is ultranormal, there exists a clopen set V which contains Z and is disjoint from $supp(m)$. Now $Z_n \bigcap (X \setminus V) \downarrow \emptyset$ and so, given $\epsilon > 0$, there exists n with $m_p(Z_n) = m_p(Z_n \bigcap (X \setminus V)) < \epsilon$. Now the result follows from the preceding Theorem.

Theorem 3.3 *Let X be ultraparacompact and $m \in M(X, E')$. Then $m \in M_\tau(X, E')$ iff $supp(m)$ is Lindelöff and, for each decreasing sequence (A_n) of clopen subsets of X with $\cap A_n$ disjoint from $supp(m)$, there exists $p \in cs(E)$ such that $m_p(A_n) \to 0$.*

Proof: Assume that $m \in M_\tau(X, E')$ and let (A_n) be as in the Theorem. The family $\mathcal{U} = \{V \in K(X) : m(A) = 0 \;\text{ if }\; A \cap V = 0\}$ is downwards directed and the family $\{A_n \bigcap (X \setminus V) : n \in \mathbb{N}, V \in \mathcal{U}\}$ is downwards directed to the empty set. Since $m \in M_\tau(X, E')$, there exists $p \in cs(E)$ such that $\lim m_p(A_n \bigcap (X \setminus V)) = 0$. Thus, given $\epsilon > 0$, there exist n and $V \in \mathcal{U}$ such that $m_p(A_n) = m_p(A_n \bigcap (X \setminus V)) < \epsilon$ and so $m_p(A_n) \to 0$. Let now $\mathcal{F}$ be a clopen cover of $supp(m)$ and let $(A_i)_{i \in I}$ be a clopen partition of X which is a refinement of the cover $\mathcal{F} \bigcup \{X \setminus supp(m)\}$. For $J \subset I$ finite, set $D_J = \bigcup_{i \notin J} A_i$. Then $D_J \downarrow \emptyset$. Thus, there exists $q \in cs(E)$ with $\lim m_q(D_J) = 0$. Given $\epsilon > 0$, there exists J_o finite with $m_q(D_{J_o}) < \epsilon$. Thus the set $M = \{i \in I : m_q(A_i) \neq 0\}$ is countable. For each $i \in M$, A_i is contained in some member of $\mathcal{F}$. It follows from this that $supp(m)$ is covered by some countable subfamily of $\mathcal{F}$ and so $supp(m)$ is Lindelöff. Conversely, assume that the condition is satisfied and let (Z_α) be a net of clopen sets which decreases to the empty set. There exists an increasing sequence (α_n) with $supp(m) \subset \bigcup_n X \setminus Z_{\alpha_n}$. There exists $p \in cs(E)$ such that $m_p(Z_{\alpha_n}) \to 0$, which implies that $\lim m_p(Z_\alpha) = 0$. Thus the result follows.

4 The Topologies β_e and β'_e

For d a continuous ultrapseudometric on X, we will denote by X_d the quotient space $X/\sim$, where $\sim$ is the equivalence relation defined by $x \sim y$ iff $d(x,y) = 0$. If $\tilde{x}_d$ is the equivalence class of x, then X_d becomes an ultrametric space under the metric $\tilde{d}(\tilde{x}_d, \tilde{y}_d) = d(x,y)$. Let $\pi_d : X \to X_d$ be the quotient map. Since π_d is continuous, we get a linear map $T_d : C_b(X_d, E) \to C_b(X, E), T_d f = f \circ \pi_d$. We define $(C_b(X,E), \beta_e)$ to be the locally convex inductive limit of the spaces $(C_b(X_d, E), \beta)$ with respect to the linear maps T_d, where d ranges over the family of all continuous ultrapseudometrics on X. Also, for $p \in cs(E)$, we define $(C_b(X,E), \beta_{e,p})$ to be the locally convex inductive limit of the spaces $(C_b(X_d,E), \beta_p)$ and $\beta'_e = \bigcup_{p \in cs(E)} \beta_{e,p}$. Note that if $p \leq q$, then $\beta_{e,p} \leq \beta_{e,q}$. It is clear that $\beta'_e \leq \beta_e$.

Theorem 4.1 *Let $h : X \to Y$ be a continuous function, where X, Y are zero-dimensional Hausdorff spaces. Then the induced linear map $S_h : C_b(Y,E) \to C_b(X,E), f \mapsto f \circ h$, is $\beta_u - \beta_u$ and $\beta'_u - \beta'_u$ continuous. Also, for $p \in cs(E)$, S_h is $\beta_p - \beta_p$ and $\beta_{u,p} - \beta_{u,p}$ continuous.*

Proof: Let W be a convex β_u-neighborhood of zero in $C_b(X,E)$. If $Q \in \Omega_u(Y)$, then there exists a clopen partition (A_i) of Y such that $\overline{A_i}^{\beta_o Y}$ is disjoint from Q for each i. If $B_i = h^{-1}(A_i)$, then (B_i) is a clopen partition of X and so the complement H in $\beta_o X$ of the set $\bigcup \overline{B_i}^{\beta_o X}$ is in $\Omega_u(X)$. Let $p \in cs(E)$ be such that W is a $\beta_{H,p}$-neighborhood of zero. Given $r > 0$, there exist $\epsilon > 0$ and a clopen subset B of X, with $\bar{B}^{\beta_o X} \cap H = \emptyset$, such that $\{g \in C_b(X,E) : \|g\|_p \leq r, \|g\|_{B,p} \leq \epsilon\} \subset W$. Since $\bar{B}^{\beta_o X} \subset \bigcup \overline{B_i}^{\beta_o X}$, there exists a finite subset J of I such that $\bar{B}^{\beta_o X} \subset \bigcup_{i \in J} \overline{B_i}^{\beta_o X}$. If $A = \bigcup_{i \in J} A_i$, then $\bar{A}^{\beta_o Y} \cap Q = \emptyset$ and

$$\{f \in C_b(Y,E) : \|f\|_p \leq r, \|f\|_{A,p} \leq \epsilon\} \subset S_h^{-1}(W).$$

This proves that $S_h^{-1}(W)$ is a $\beta_{Q,p}$-neighborhood of zero and the $\beta_u - \beta_u$ continuity of S_h follows. The proof of the $\beta'_u - \beta'_u$ continuity is analogous. The proof of the $\beta_p - \beta_p$ continuity is similar to the proof of the $\beta - \beta$ continuity in Theorem 2.21 while the proof of the $\beta_{u,p} - \beta_{u,p}$ continuity is analogous to the one of the $\beta_u - \beta_u$ continuity. Hence the result follows.

Theorem 4.2 *$\beta_u \leq \beta_e$ and $\beta'_u \leq \beta'_e$*

Proof: For an ultrametrizable space Y we have that $\Omega(Y) = \Omega_u(Y)$ and so $\beta = \beta_u$ and $\beta_p = \beta_{u,p}$ for each $p \in cs(E)$. In view of the preceding Theorem, for each continuous ultrapseudometric d on X, T_d is $\beta - \beta_u$ continuous and so $\beta_u \leq \beta_e$. Also, T_d is $\beta_p - \beta_{u,p}$ continuous, which implies that $\beta'_u \leq \beta'_e$.

Theorem 4.3 *Let $h : X \to Y$ be a continuous function, where X, Y are zero-dimensional Hausdorff spaces. Then, the induced linear map S_h is $\beta_e - \beta_e$ and $\beta'_e - \beta'_e$ continuous.*

Proof: Let d be a continuous ultrapseudometric on Y and define d_1 on $X \times X$ by $d_1(x,y) = d(h(x), h(y))$. Then d_1 is continuous. Let $\psi : X_{d_1} \to Y_d, \tilde{x}_{d_1} \mapsto \tilde{h}(x)_d$. Then ψ is well defined and continuous. Let $S_\psi : C_b(Y_d, E) \to C_b(X_{d_1}, E)$ be the induced linear map. Then $T_{d_1} \circ S_\psi = S_h \circ T_d$. Since T_{d_1} is $\beta - \beta_e$ is continuous and S_ψ $\beta - \beta$ continuous, it follows that $S_h \circ T_d$ is $\beta - \beta_e$ continuous. This clearly proves that S_h is $\beta_e - \beta_e$ continuous. The proof of the $\beta_{e'} - \beta'_e$ continuity of S_h is analogous.

Theorem 4.4 *If E is metrizable, then $\beta_e \leq \beta_1, \beta_e' \leq \beta_1'$ and $\beta_{e,p} \leq \beta_{1,p}$ for each $p \in cs(E)$.*

Proof: Assume that there exists a convex β_e-neighborhood W of zero and a $Q \in \Omega_1$ such that W is not a β_Q-neighborhood of zero. Let (p_n) be an increasing sequence of continuous seminorms on E generating its topology and let $h \in C_{rc}(X)$ be such that $Q = \{x \in \beta_o X : h^{\beta_o}(x) = 0\}$. For each positive integer n, let $A_n = \{x \in X : |h(x)| \geq 1/n\}$. Then $\overline{A_n}^{\beta_o X} = \{x \in \beta_o X : |h^{\beta_o}(x)| \geq 1/n\}$. Since W is not a β_{Q,p_n}-neighborhood of zero, there exists $r_n > 0$ such that, for each clopen B, with $\bar{B}^{\beta_o X}$ disjoint from Q, and each $\epsilon > 0$, there exists $f \in C_b(X, E)$ with $\|f\|_{p_n} \leq r_n, \|f\|_{B,p_n} \leq \epsilon, f \notin W$. Hence, for each positive integer k, there exists $f_{nk} \in C_b(X, E), f_{nk} \notin W, \|f_{nk}\|_{p_n} \leq r_n, \|f_{nk}\|_{A_k,p_n} \leq 1/k$. Let $\alpha_{nk} > \max\{\|f_{ij}\|_{p_n} : 1 \leq i \leq n, 1 \leq j \leq k\}$ and define

$$d(x,y) = \max\left\{|h(x) - |h(y)|, \ \sup_{n,k} \frac{1}{kn\alpha_{nk}}\left[\max_{1\leq i\leq n, 1\leq j\leq k} p_n(f_{ij}(x) - f_{ij}(y))\right]\right\}.$$

Then d is a continuous ultra-pseudometric on X and so $T_d^{-1}(W)$ is a β-neighborhood of zero in $C_b(X_d, E)$. The set $H = \pi_d^{\beta_o}(Q)$ is disjoint from X_d. In fact, assume that $\pi_d^{\beta_o}(x) = \pi_d(a)$ for some $a \in X, x \in Q$. There exists a net (x_δ) in X converging to x and so $\pi_d(x_\delta) \to \pi_d(a)$, i.e. $d(x_\delta, a) \to 0$. Since $h(a) \neq 0$, there exists δ_o such that $d(x_\delta, a) < |h(a)|$ and thus $|h(x_\delta)| = |h(a)|$, which contradicts the fact that $h(x_\delta) \to h^{\beta_o}(x) = 0$. So, H is disjoint from X_d and therefore $T_d^{-1}(W)$ is a β_{H,p_n}-neighborhood of zero for some n. There are an $\epsilon > 0$ and a clopen subset A of X_d, with $\bar{A}^{\beta_o X_d} \cap H = \emptyset$, such that $\{g \in C_b(X_d, E) : \|g\|_{p_n} \leq r_n, \|g\|_{A,p_n} \leq \epsilon\} \subset T_d^{-1}(W)$. Let $B = \pi_d^{-1}(A)$. Then $\bar{B}^{\beta_o X}$ is disjoint from Q and hence $B = \pi_d^{-1}(A) \subset \bigcup_n\{x : |h^{\beta_o}(x)| \geq 1/n\} = \bigcup_n \overline{A_n}^{\beta_o X}$. Choose $k > 1/\epsilon$ such that $B = \pi_d^{-1}(A) \subset \overline{A_k}^{\beta_o X}$. The function $g : X_d \to E, g(\tilde{x}_d) = f_{nk}(x)$ is well defined, continuous and $T_d g = f_{nk}$. Since $\|g\|_{p_n} \leq r_n$ and $\|g\|_{A,p_n} = \|f_{nk}\|_{B,p_n} \leq 1/k \leq \epsilon$, we have that $g \in T_d^{-1}(W)$ and so $f_{nk} \in W$, a contradiction. This proves that $\beta_e \leq \beta_1$. Suppose next that there exists a convex $\beta_{e,p}$-neighborhood W of zero which is not a $\beta_{Q,p}$-neighborhood for some $Q \in \Omega_1$. There exists $r > 0$ such that, for every clopen subset B of X , whose closure in $\beta_o X$ is disjoint from Q, and any $\epsilon > 0$, there exists $f \in C_b(X, E)$ with $\|f\|_p \leq r, \|f\|_{B,p} \leq \epsilon, f \notin W$. Let $h \in C_{rc}(X)$ be such that $Q = \{x \in \beta_o X : h^{\beta_o}(x) = 0\}$ and let $A_n = \{x : |h(x)| \geq 1/n\}$. For each n, there exists an $f_n \in C_b(X, E)$ with $\|f_n\|_p \leq r, \|f_n\|_{A_n,p} \leq 1/n, f_n \notin W$. Let (p_n) be an increasing sequence of continuous seminorms on E generating its topology. Choose $\alpha_n > \max\{\|f_k\|_{p_n} : 1 \leq k \leq n\}$ and define

$$d(x,y) = \max\left\{|h(x) - h(y)|, \sup_n \frac{1}{n\alpha_n}\left[\max_{1\leq k\leq n} p_n(f_k(x) - f_k(y))\right]\right\}.$$

Then d is a continuous ultra-pseudometric on X and so $T_d^{-1}(W)$ is a β_p-neighborhood of zero. Since $H = \pi_d^{\beta_o}(Q)$ is disjoint from X_d, there exist a clopen subset A of X_d, with $\bar{A}^{\beta_o X_d} \cap H = \emptyset$, and $\epsilon > 0$ such that

$$\{g \in C_b(X_d, E) : \|g\|_p \leq r, \|g\|_{A,p} \leq \epsilon\} \subset T_d^{-1}(W).$$

Choose $n > 1/\epsilon$ such that $\bar{B}^{\beta_o X} \subset \overline{A_n}^{\beta_o X}$, where $B = \pi_d^{-1}(A)$. The function $g : X_d \to E, g(\tilde{x}_d) = f_n(x)$, is well defined and continuous. Since $\|g\|_p = \|f_n\|_p \leq r$ and $\|g\|_{A,p} = \|f_n\|_{B,p} \leq 1/n \leq \epsilon$ we have that $f_n = T_d g \in W$, a contradiction. This proves that $\beta_{e,p} \leq \beta_{1,p}$, for each $p \in cs(E)$, which implies that $\beta_e' \leq \beta_1'$. This completes the proof.

Theorem 4.5 *Assume that β_e is coarser than τ_u, e.g. when E is metrizable. Then, on each uniformly bounded equicontinuous subset H of $C_b(X,E)$, β_e coincides with the topology τ_s of pointwise convergence.*

Proof: We may assume that H is absolutely convex. Let W be a convex β_e-neighborhood of zero. Since β_e is coarser than τ_u, there exists $p \in cs(E)$ such that $W_1 = \{f \in C_b(X,E) : \|f\|_p \leq 1\} \subset W$. Consider the continuous ultra-pseudometric d on X defined by $d(x,y) = \sup_{f \in H} p(f(x) - f(y))$ and let $(A_i)_{i \in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $d(x,y) \leq 1$. Let $V = T_d^{-1}(W)$ and $Q = \beta_o X \setminus \bigcup_{i \in I} \overline{A_i}^{\beta_o X}$. Then $D = \pi_d^{\beta_o}(Q)$ is disjoint from X_d. Let $q \in cs(E), q \geq p$, be such that V is a $\beta_{D,q}$-neighborhood of zero. If $r > \sup_{f \in H} \|f\|_q$, then there exist $\epsilon_1 > 0$ and a clopen set B in X_d, with $\bar{B}^{\beta_o X_d} \cap D = \emptyset$, such that $\{g \in C_b(X_d,E) : \|g\|_q \leq r, \|g\|_{D,q} \leq \epsilon_1\} \subset V$. If $A = \pi_d^{-1}(B)$, then $\bar{A}^{\beta_o X} \subset \bigcup \bar{A}_i^{\beta_o X}$ and so $\bar{A}^{\beta_o X} \subset \bigcup_{i \in J} \bar{A}_i^{\beta_o X}$ for some finite subset J of I. Choose $x_i \in A_i$, for each $i \in I$, and let $W_2 = \{f \in C_b(X,E) : p(f(x_i)) \leq \epsilon_1 \quad \text{for} \quad i \in J\}$. Then $W_2 \cap H \subset W$. Indeed, let $f \in W_2 \cap H$ and let $f^* = \sum_{i \in I} \chi_{A_i} f(x_i)$. The function $h : X_d \to E, h(\tilde{x}_d) = f^*(x)$ is well defined, bounded, continuous and $T_d h = f^*$. Moreover, $\|f^*\|_q \leq r$. If $\tilde{x}_d \in B$, then $x \in A$ and so $x \in A_i$, for some $i \in J$, which implies that $h(\tilde{x}_d) = f(x_i)$. Thus $\|h\|_{B,q} \leq \epsilon_1$ and therefore $h \in V$, i.e $f^* \in W$. Also $\|f - f^*\|_p \leq 1$ and so $f - f^* \in W$. Thus $f \in W$. This proves that the topology induced on H by τ_s is finer than the one induced by β_e and the proof is complete since τ_s is coarser that β_e.

The proof of the following Theorem is analogous to the one of the preceding Theorem.

Theorem 4.6 *For $p \in cs(E)$, let τ_p be the topology on $C_b(X,E)$ generated by the seminorm $\|.\|_p$. If $\beta_{e,p}$ is coarser than τ_p, then on τ_p-bounded p-equicontinuous subsets of $C_b(X,E)$, $\beta_{e,p}$ coincides with the topology generated by the seminorms $f \mapsto p(f(x)), x \in X$.*

Theorem 4.7 *Assume that τ_u is finer than β_e and let W be a convex β_e-neighborhood of zero. Then, for each $f \in C_b(X,E)$, there are pairwise disjoint clopen sets $A_1, \ldots A_n$ in X and $x_k \in A_k$ such that $f - \sum_{k=1}^n \chi_{A_k} f(x_k) \in W$.*

Proof: We may assume that W is convex. Since τ_u is finer than β_e , there exists $p \in cs(E)$ such that $W_1 = \{g \in C_b(X,E) : \|g\|_p \leq 1\} \subset W$. Let $(A_i)_{i \in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $p(f(x) - f(y)) \leq 1$. Let $h_i = \chi_{A_i}, x_i \in A_i, f^* = \sum_{i \in I} h_i f(x_i)$. Then $f - f^* \in W_1$. The ultra-pseudometric $d(x,y) = \sup_i |h_i(x) - h_i(y)|$ is continuous and so $V = T_d^{-1}(W)$ is a β-neighborhood of zero in $C_b(X_d,E)$. Let $Q = \beta_o X \setminus \bigcup_i \overline{A_i}^{\beta_o X}$ and $D = \pi_d^{\beta_o}(Q)$. Then D is disjoint from X_d and hence there exists $q \in cs(E)$ such that V is a $\beta_{D,q}$-neighborhood of zero. There are $\epsilon > 0$ and a clopen subset A of X_d, with $\bar{A}^{\beta_o X_d} \cap D = \emptyset$, such that

$$\{g \in C_b(X_d,E) : \|g\|_q \leq \|f\|_q, \|g\|_{A,q} \leq \epsilon\} \subset V.$$

If $B = \pi_d^{-1}(A)$, then $\bar{B}^{\beta_o X} \subset \bigcup_i \overline{A_i}^{\beta_o X}$ and so $\bar{B}^{\beta_o X} \subset \bigcup_{i \in J} \overline{A_i}^{\beta_o X}$ for some finite subset J of I. Let $g = \sum_{i \in J} h_i f(x_i), g_1 = f^* - g$. The function $\bar{g_1} : X_d \to E, \bar{g_1}(\tilde{x}_d) = g_1(x)$ is well defined, continuous and $T_d \bar{g_1} = g_1$. Since $\|\bar{g_1}\|_q \leq \|f\|_q$ and $\bar{g_1} = 0$ on A, it follows that $\bar{g_1} \in V$ and so $g_1 \in W$. Finally, $f - g = (f - f^*) + g_1 \in W$, which completes the proof.

5 The Dual Spaces of $(C_b(X,E),\beta_u)$ and $(C_b(X,E),\beta'_u)$

We will denote by $M_u(X,E')$ the space of all $m \in M(X,E')$ with the following property: For each decreasing net (A_δ) of clopen subsets of X with $\bigcap_\delta \overline{A_\delta}^{\beta_o X} \in \Omega_u$, there exists $p \in cs(E)$ such that $m_p(X) < \infty$ and $m_p(A_\delta) \to 0$.

Theorem 5.1 *If $m \in M_u(X,E')$, then every $f \in C_b(X,E)$ is m-integrable and the linear functional $u_m(f) = \int f dm$, $f \in C_b(X,E)$, is β_u-continuous.*

Proof: There exists a $p \in cs(E)$ such that $m_p(X) \le 1$. Let $f \in C_b(X,E)$ and $\epsilon > 0$. Without loss of generality, we may assume that $\|f\|_p \le 1$. Let $(A_i)_{i\in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $p(f(x) - f(y)) \le \epsilon$. For $J \subset I$ finite, set $B_J = \bigcup_{i\notin J} A_i$. Then (B_J) is a decreasing net of clopen sets with $Q = \bigcap \overline{B_J}^{\beta_o X} = \beta_o X \setminus \bigcup_{i\in I} \overline{A_i}^{\beta_o X}$ and hence $Q \in \Omega_u$. Thus, there exists $q \in cs(E), q \ge p$, such that $m_q(B_J) \to 0$. We now finish the proof by using an argument analogous to the one used in the proof of Theorem 2.4.

Argueing as in the proof of Theorem 2.5, we get the following

Theorem 5.2 *The map $m \mapsto u_m$, from $M_u(X,E')$ to $(C_b(X,E),\beta_u)'$, is an algebraic isomorphism.*

Next we will look at the dual space of $(C_b(X,E),\beta'_u)$. For $p \in cs(E)$, let $\mathcal{M}_{u,p}(X.E')$ be the space of all $m \in M_p(X,E')$ with the following property: For each decreasing net (A_δ) of clopen subsets of X, with $\bigcap_\delta \overline{A_\delta}^{\beta_o X} \in \Omega_u$, we have that $m_p(A_\delta) \to 0$. Let

$$\mathcal{M}_u(X,E') = \bigcup_{p\in cs(E)} \mathcal{M}_{u,p}(X.E').$$

Clearly $\mathcal{M}_u(X,E' \subset M_u(X,E')$. Using arguments analogous to the ones used in the proofs of Theorems 2.4 and 2.5, we get the following

Theorem 5.3 *a) For each $m \in \mathcal{M}_{u,p}(X.E'), u_m$ is $\beta_{u,p}$-continuous and the map $m \mapsto u_m$, from $\mathcal{M}_{u,p}(X,E')$ to $(C_b(X,E),\beta_{u,p})'$, is an algebraic isomorphism.*
b) $\mathcal{M}_u(X,E')$ is algebraically isomorphic to $(C_b(X,E),\beta'_u)'$ via the isomorphism $m \mapsto u_m$.

Theorem 5.4 *Let $m \in M(X,E')$. Then 1) $m \in M_u(X,E')$ iff the following condition is satisfied: For each clopen partition $(A_i)_{i\in I}$ of X, there exists $p \in cs(E)$, with $m_p(X) < \infty$, such that, for each $\epsilon > 0$, there exists a finite subset J of I with $m_p(\bigcup_{i\notin J} A_i) \le \epsilon$.*
2) $m \in \mathcal{M}_{u,p}(X,E')$ iff $m_p(X) < \infty$ and, for each clopen partition $(A_i)_{i\in I}$ of X and each $\epsilon > 0$, there exists a finite subset J of I such that $m_p(\bigcup_{i\notin J} A_i) \le \epsilon$.

Proof: 1) Assume that $m \in M_u(X,E')$ and let $(A_i)_{i\in I}$ be a clopen partition of X. For $J \subset I$ finite, set $B_J = \bigcup_{i\notin J} A_i$. Then $Q = \bigcap \overline{B_J}^{\beta_o X} \in \Omega_u$ and hence there exists $p \in cs(E)$, with $m_p(X) < \infty$, such that $m_p(B_J) \to 0$.
Conversely, assume that the condition is satisfied and let (B_δ) be a decreasing net of clopen sets with $\bigcap \overline{B_\delta}^{\beta_o X} = D \in \Omega_u$. There exists a clopen partition $(A_i)_{i\in I}$ of X such that each $\overline{A_i}^{\beta_o X}$ is disjoint from D. Let $p \in cs(E)$ be as in the condition. Given $\epsilon > 0$, there

exists $J \subset I$ finite such that $m_p(D_J) \leq \epsilon$, where $D_J = \bigcup_{i \notin J} A_i$. If $M_J = X \setminus D_J$, then $\overline{M_J}^{\beta_o X} = \bigcup_{i \in J} \overline{A_i}^{\beta_o X} \subset \bigcup_\delta \beta_o X \setminus \overline{B_\delta}^{\beta_o X}$. There exists δ such that $\overline{M_J}^{\beta_o X} \subset \beta_o X \setminus \overline{B_\delta}^{\beta_o X}$. Thus $\overline{B_\delta}^{\beta_o X} \subset \overline{D_J}^{\beta_o X}$ and so $m_p(B_\delta) \leq \epsilon$, which proves 1).
2) The proof is analogous to that of 1).

Remark 5.5 *In view of the preceding Theorem, $\mathcal{M}_u(X, E')$ coincides with the space $M'_u(X, E')$ introduced in [3]. In the same paper it was shown that the dual space of $(C_{rc}(X, E), \beta'_u)$ is $M'_u(X, E')$.*

Theorem 5.6 *For a subset H of $M_u(X, E')$, the following are equivalent:*
(1) H is β_u-equicontinuous.
(2) For each decreasing net (A_δ) of clopen subsets of X with $\bigcap_\delta \overline{A_\delta}^{\beta_o X} \in \Omega_u$, there exists $p \in cs(E)$ such that $\sup_{m \in H} m_p(X) < \infty$ and $m_p(A_\delta) \to 0$ uniformly for $m \in H$.
(3) For each clopen partition $(A_i)_{i \in I}$ of X, there exists $p \in cs(E)$, with $\sup_{m \in H} m_p(X) < \infty$, such that, for each $\epsilon > 0$, there exists a finite subset J of I with $m_p(\bigcup_{i \notin J} A_i) \leq \epsilon$ for all $m \in H$.

Proof: The equivalence of (2) and (3) can be proved using an argument analogous to the one used in the proof of Theorem 5.4.
(1)⇒ (2). Let H^o be the polar of H in $C_b(X, E)$. Since $\beta_u \leq \tau_u$, there exists $q \in cs(E)$ such that $\{f \in C_b(X, E) : \|f\|_q \leq 1\} \subset H^o$. It follows from this that $\sup_{m \in H} m_q(X) < \infty$. Let now (A_δ) be a decreasing net of clopen sets with $Q = \bigcap_\delta \overline{A_\delta}^{\beta_o X} \in \Omega_u$. Since H^o is a β_u-neighborhood of zero, there exist $p \in cs(E), p \geq q$, and $h \in C_Q$ such that $W_2 = \{f \in C_b(X, E) : \|hf\|_p \leq 1\} \subset H^o$. We will prove that $\lim_\delta m_p(A_\delta) = 0$ uniformly for $m \in H$. So let μ be a non-zero element of $\mathbb{K}$. The set $D = \{x \in \beta_o X : |h^{\beta_o}(x)| \leq |\mu|\}$ contains Q and so it contains some $\bar{A}_\delta^{\beta_o X}$. If now A is a clopen subset of X contained in A_δ and $s \in E$ with $p(s) \leq 1$, then $\mu^{-1} \chi_A s \in W_2$ and thus $|m(A)s| \leq |\mu|$ for all $m \in H$. If $\lambda \in \mathbb{K}, |\lambda| > 1$, then $m_p(A_\delta) \leq |\mu\lambda|$, which proves that $m_p(A_\delta) \to 0$ uniformly for $m \in H$.
(2)⇒ (1). Let $Q \in \Omega_u$. There exists a decreasing net (A_δ) of clopen subsets of X such that $\bigcap \bar{A}_\delta^{\beta_o X} = Q$. Let p be as in (2) and let $d > \sup_{m \in H} m_p(X)$. Given $r > 0$, there exists a δ such that $m_p(A_\delta) < 1/r$ for all $m \in H$. If $B = X \setminus A_\delta$, then $\bar{B}^{\beta_o X}$ is disjoint from Q and $\{f \in C_b(X, E) : \|f\|_p \leq r, \|f\|_{B,p} \leq 1/d\} \subset H^o$, which proves that H^o is a $\beta_{Q,p}$-neighborhood of zero. It follows that H^o is a β_u-neighborhood and so H is β_u-equicontinuous.

The proof of the following Theorem is analogous to the one of the preceding Theorem.

Theorem 5.7 *For a subset H of $\mathcal{M}_u(X, E')$ and $p \in cs(E)$ the following are equivalent:*
(1) H is $\beta_{u,p}$-equicontinuous.
(2) $\sup_{m \in H} m_p(X) < \infty$ and for each decreasing net (A_δ) of clopen subsets of X with $\bigcap_\delta \overline{A_\delta}^{\beta_o X} \in \Omega_u$, we have that $m_p(A_\delta) \to 0$ uniformly for $m \in H$.
(3) $\sup_{m \in H} m_p(X) < \infty$ and for each clopen partition $(A_i)_{i \in I}$ of X and each $\epsilon > 0$, there exists a finite subset J of I such that $m_p(\bigcup_{i \notin J} A_i) \leq \epsilon$ for each $m \in H$.

6 The Dual Spaces of $(C_b(X,E),\beta_e)$ and $(C_b(X,E),\beta'_e)$

Theorem 6.1 *Assume that β_e is coarser than τ_u and let u be a β_e-continuous linear form on $C_b(X,E)$. Let $m \in M(X,E')$ be such that $u(f) = \int f dm$ for all $f \in C_{rc}(X,E)$. For $f \in C_b(X,E)$ and $A \in K(X)$, set $|m|_f(A) = \sup\{|m(B)f(x)| : x \in X, B \in K(X), B \subset A\}$. If $(A_i)_{i\in I}$ is a clopen partition of X, then:*
(1) For each $g \in C_b(X,E)$ of the form $g = \sum \chi_{A_i}s_i$, $s_i \in E$, we have that $u(g) = \sum_i m(A_i)s_i$.
(2) For each $\epsilon > 0$, the set $I_\epsilon = \{i \in I : |m|_f(A_i) \geq \epsilon\}$ is finite.
(3) If $x_i \in A_i$, then the function $f^* = \sum_i \chi_{A_i}f(x_i)$ is m-integrable.

Proof: (1) For $J \subset I$ finite, let $h_J = \sum_{i\in J}\chi_{A_i}s_i$. The set $\{h_J : J \text{ finite}\}$ is uniformly bounded, equicontinuous and $h_J \to g$ pointwise. By Theorem 4.5, we have that $\sum_{i\in J} m(A_i)s_i = u(h_J) \to u(g)$.
(2) For each i, there exist a clopen subset B_i of A_i and $x_i \in X$ such that $|m(B_i)f(x_i)| \geq |m|_f(A_i)/2$. The set $B = \cup_i B_i$ is clopen. Using (1), we get that $u(\sum_{i\in I}\chi_{B_i}f(x_i)) = \sum_{i\in I} m(B_i)f(x_i)$. There exists a finite subset J of I such that $|m(B_i)f(x_i)| < \epsilon/2$ if $i \notin J$. For such i, we have $|m|_f(A_i) < \epsilon$ and so $i \notin I_\epsilon$.
(3) Let $\epsilon > 0$ and $D = \bigcup_{i\notin I_\epsilon} A_i$. Let $\{D_1,\dots,D_N\}$ be a clopen partition of X, which is a refinement of $\{A_i : i \in I_\epsilon\} \cup \{D\}$, and let $y_k \in D_k$. We may assume that $\bigcup_{k=1}^r D_k = \bigcup_{i\in I_\epsilon} A_i$. Then $\sum_{k=1}^r m(D_k)f^*(y_k) = \sum_{i\in I_\epsilon} m(A_i)f(x_i)$. Let A be a clopen subset of D and $z \in A$. Using (1), we get that $m(A)f^*(z) = \sum_{i\notin I_\epsilon} m(A\cap A_i)f^*(z)$. But, for $i \notin I_\epsilon$, $|m(A\cap A_i)f^*(z)| \leq |m|_f(A_i) < \epsilon$. Thus $|m(A)f^*(z)| \leq \epsilon$ and so $|\sum_{k=r+1}^N m(D_k)f^*(y_k)| \leq \epsilon$, which implies that $|\sum_{k=1}^N m(D_k)f^*(y_k) - \sum_{i\in I_\epsilon} m(A_i)f(x_i)| \leq \epsilon$. It clearly follows that f^* is m-integrable.

Theorem 6.2 *Assume that β_e is coarser than τ_u and let u and m be as in the preceding Theorem. Then: (1) Every $f \in C_b(X,E)$ is m-integrable.*
(2) If $(A_i)_{i\in I}$ is a clopen partition of X and $(s_i)_{i\in I}$ a bounded family in $\mathbb{K}$, then for $g = \sum \chi_{A_i}s_i$, we have that $\int g dm = u(g) = \sum_i m(A_i)s_i$.
(3) $u(f) = \int f dm$ for each $f \in C_b(X,E)$.

Proof: (1) Let $\epsilon > 0$ and let $p \in cs(E)$ be such that $m_p(X) \leq 1$. Let $(A_i)_{i\in I}$ be the clopen partition of X which corresponds to the equivalence relation $x \sim y$ iff $p(f(x) - f(y)) \leq \epsilon$. Let $x_i \in A_i$, $f^* = \sum \chi_{A_i}f(x_i)$. Since f^* is m-integrable, there exists a clopen partition $\{Z_1,\dots,Z_n\}$ of X and $z_k \in Z_k$ such that, for each clopen partition $\{D_1,\dots,D_N\}$ of X, which is a refinement of $\{Z_1,\dots,Z_n\}$, and any $y_j \in D_j$, we have $|\sum_{k=1}^n m(Z_k)f^*(z_k) - \sum_{j=1}^N m(D_j)f^*(y_j)| \leq \epsilon$. Since, for $h = f - f^*$, we have $|m(A)h(z)| \leq \epsilon$ for all $A \in K(X)$ and all $z \in X$, we have that $|\sum_{j=1}^N m(D_j)f(y_j) - \sum_{k=1}^n m(Z_k)f(z_k)| \leq \epsilon$, which proves that f is m-integrable.
(2) Given $\epsilon > 0$, there exists a finite subset J_o of I such that $|m|_g(A_i) < \epsilon$ if $i \notin J_o$. If the clopen set A is disjoint from $D = \bigcup_{i\in J_o} A_i$, then $|m(A)g(x)| \leq \epsilon$ for all $x \in X$. Also there exists a finite subset J of I containing J_o such that $|u(g) - \sum_{i\in J} m(A_i)s_i| < \epsilon$. If $h = \sum_{i\notin J}\chi_{A_i}s_i$, then $|\int h dm| \leq \epsilon$ and so $|\int g dm - \sum_{i\in J} m(A_i)s_i| \leq \epsilon$, which implies that $|\int g dm - u(g)| \leq \epsilon$.
(3) Let $f \in C_b(X,E)$ and $\epsilon > 0$. There exists $p \in cs(E)$ such that $m_p(X) \leq 1$ and $W_1 = \{g \in C_b(X,E) : \|g\|_p \leq 1\} \subset \{g : |u(g)| \leq 1\}$. Let $(A_i)_{i\in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $p(f(x) - f(y)) \leq \epsilon$. Choose $x_i \in A_i$ and let $f^* = \sum \chi_{A_i}f(x_i)$, $h = f - f^*$. There exists a finite subset J of I such that

$|u(f^*)-\sum_{i\in J} m(A_i)f(x_i)| < \epsilon$ and $|m|_f(A_i) < \epsilon$ if $i \notin J$. If $D = \bigcup_{i\in J} A_i$ and $Z = X\setminus D$, then $|\int_Z f^* dm| \le \epsilon$ and $\int_D f^* dm = \sum_{i\in J} m(A_i)f(x_i)$ and so $|\int f^* dm - \sum_{i\in J} m(A_i)f(x_i)| \le \epsilon$. Thus $|\int f^* dm - u(f^*)| \le \epsilon$. If $|\lambda| > 1$, then $|u(h)| \le \epsilon|\lambda|$ and $|\int h dm| \le \epsilon$. It follows that $|\int f dm - u(f)| \le \epsilon|\lambda|$, which clearly completes the proof.

By the above Theorem, we have the following

Theorem 6.3 *If β_e is coarser than τ_u, then there exists a subspace $M_e(X,E')$ of $M(X,E')$ such that every $f \in C_b(X,E)$ is m-integrable for every $m \in M_e(X,E')$ and the map $m \mapsto u_m$, from $M_e(X,E')$ to $(C_b(X,E),\beta_e)'$, is an algebraic isomorphism.*

Conjecture 6.4 *If β_e is coarser than τ_u, then $M_e(X,E') = M_u(X,E')$.*

For $p \in cs(E)$, d a continuous ultra-pseudometric on X and A a d-clopen subset of X, we define

$$|m|_{d,p}(A) = \sup\left\{\frac{|m(B)s|}{p(s)} : p(s) \neq 0, B \subset A, B \;\; d-\text{clopen}\right\}.$$

Also, for $Y \subset X$, we define $|m|^*_{d,p}(Y)$ to be the infimum of all $\sup_n m_{d,p}(A_n)$, where the infimum is taken over all sequences (A_n) of d-clopen sets which cover Y. We will need the following

Theorem 6.5 *Let (Z,d) be an ultrametric space and assume that E has the countable neighborhood property. If $m \in M_\tau(Z,E')$, then there exists $p \in cs(E)$, with $m_p(Z) < \infty$, and a d-closed, d-separable subset G of Z such that $|m|^*_{d,p}(Z\setminus G) = 0$.*

Proof: For $Y \subset Z$ finite and $\epsilon > 0$, set $N(Y,\epsilon) = \{x \in Z : d(x,Y) \le \epsilon\}$. The family $\{Z\setminus N(Y,\epsilon) : Y \text{ finite}\}$ is downwards directed to the empty set. Since $m \in M_\tau(Z,E')$, there exists $q \in cs(E)$, with $m_q(Z) < \infty$, such that $\lim_Y |m|_{d,q}(Z\setminus N(Y,\epsilon) = 0$. Hence, there exists an increasing sequence (p_n) in $cs(E)$ such that $\lim_Y |m|_{d,p_n}(Z\setminus N(Y,1/n) = 0$ for all n. Since E has the countable neighborhood property, there exist $p \in cs(E)$ and a sequence (μ_n) of non-zero elements of $\mathbb{K}$ such that $p \ge |\mu_n|p_n$ for all n. For each k, choose an increasing sequence $(Y_{k,n})_n$ of finite subsets of Z such that $|m|_{d,p_k}(Z\setminus N(Y_{k,n},1/k)) < |\mu_k|/n$ for all n. Let $D_n = \bigcup_k (Z\setminus N(Y_{k,n},1/k)), M = \bigcup_n Z\setminus D_n$ and $G = \bar{M}$. We have

$$|m|_{d,p}(Z\setminus N(Y_{k,n},1/k)) \le |\mu_k|^{-1}|m|_{d,p_k}(Z\setminus N(Y_{k,n},1/k)) \le 1/n.$$

Thus $|m|^*_{d,p}(Z\setminus G) = 0$. Also, G is d-separable. Indeed, let $x \in G$ and $\epsilon > 0$. There exists $y \in M$ with $d(x,y) < \epsilon$. Let n be such that $y \notin D_n$. Choose $k > 1/\epsilon$. Since $y \in N(Y_{k,n},1/k)$, there exists $z \in Y_{k,n}$ with $d(z,y) \le 1/k < \epsilon$ and so $d(x,z) < \epsilon$. It follows that G is contained in $\bar{L}$, where $L = \bigcup_{n,k} Y_{k,n}$. Since $\bar{L}$ is separable, its subspace G is also separable. This completes the proof.

Let $M_s(X,E')$ be the space of all $m \in M(X,E')$ with the following property: For each continuous ultra-pseudometric d on X, there exist $p \in cs(E)$, with $m_p(X) < \infty$, and a d-closed, d-separable subset G of X such that $|m|^*_{d,p}(Z\setminus G) = 0$.

Theorem 6.6 *If β_e is coarser than τ_u and E has the countable neighborhhod property, then every $m \in M_e(X,E')$ is in $M_s(X,E')$.*

Proof: Let d be a given continuous ultra-pseudometric on X. Since u_m is β_e-continuous, $T_d^* u_m$ is β-continuous on $C_b(X_d, E)$. Thus, there is $\mu \in M_\tau(X_d, E')$ such that $\int g d\mu = \int (T_d g) dm$ for all $g \in C_b(X_d, E)$. By the preceding Theorem, there exists a $\tilde{d}$-closed, $\tilde{d}$-separable subset Z of X_d such that $|\mu|^*_{\tilde{d},p}(X_d \setminus Z) = 0$. The set $G = \pi_d^{-1}(Z)$ is d-closed, d-separable and $|m|^*_{d,p}(X \setminus G) = 0$.

We will look next at the dual space of $(C_b(X, E), \beta'_e)$. For $p \in cs(E)$, we denote by $\mathcal{M}_{s,p}(X, E')$ the space of all $m \in \mathcal{M}_{\sigma,p}(X, E')$ with the following property: For each continuous ultra-pseudometric d on X, there exists a d-closed, d-separable subset G of X such that $|m|^*_{d,p}(X \setminus G) = 0$. Let $\mathcal{M}_s(X, E') = \bigcup \mathcal{M}_{s,p}(X, E')$. For the proof of the following theorem we may use an argument analogous to the one used in the proof of Theorem 6.5. Note that in the next Theorem we don't need to assume that E has the countable neighborhood property.

Theorem 6.7 *Let (Z, d) be an ultrametric space and let $p \in cs(E)$ and $m \in \mathcal{M}_{\tau,p}(Z, E')$. Then there exists a d-closed, d-separable subset G of Z such that $|m|^*_{d,p}(Z \setminus G) = 0$*

Theorem 6.8 *If E is metrizable, then $\mathcal{M}_{s,p}(X, E')$ is algebraically isomorphic to the dual space of $(C_b(X, E), \beta_{e,p})$.*

Proof: Let u be in the dual space of $(C_b(X, E), \beta_{e,p})$. By Theorem 6.3, there exists $m \in M_e(X, E')$ such that $u(f) = \int f dm$ for all $f \in C_b(X, E)$. Since $\beta_{e,p} \leq \beta_{1,p}$, m is in $\mathcal{M}_{\sigma,p}(X, E')$ (see Theorem 2.17). We will show that $m \in \mathcal{M}_{s,p}(X, E')$. So, let d be a continuous ultra-pseudometric on X. Since T_d is $\beta_p - \beta_{e,p}$ continuous, $T_d^* u$ is β_p-continuous on $C_b(X_d, E)$ and so there exists $\mu \in \mathcal{M}_{\tau,p}(X_d, E')$ such that $\int g d\mu =< T_d g, u >$ for all $g \in C_b(X_d, E)$. In view of the preceding Theorem, there exists a $\tilde{d}$-closed, $\tilde{d}$-separable subset Z of X_d with $|\mu|^*_{\tilde{d},p}(X_d \setminus Z) = 0$. The set $G = \pi_d^{-1}(Z)$ is d-closed, d-separable and $|m|^*_{d,p}(X \setminus G) = 0$, which proves that $m \in \mathcal{M}_{s,p}(X, E')$. Conversely, let $m \in \mathcal{M}_{s,p}(X, E')$ and let d be a continuous ultra-pseudometric on X. Define $\mu = \mu_d : K(X_d) \to E', \mu(A) = m(\pi_d^{-1}(A))$. Then $\mu \in \mathcal{M}_{\sigma,p}(X_d, E')$ since $m \in \mathcal{M}_{\sigma,p}(X, E')$.
Claim I: $\mu \in \mathcal{M}_{\tau,p}(X_d, E')$. Indeed, let (V_δ) be a net of clopen subsets of X_d which decreases to the empty set. By our hypothesis, there is a d-closed, d-separable subset G of X such that $|m|^*_{d,p}(Z \setminus G) = 0$. Given $\epsilon > 0$, there is an increasing sequence (Z_n) of d-clopen subsets of X covering $X \setminus G$ and such that $|m|_{d,p}(Z_n) < \epsilon$ for all n. If $M = \pi_d(G)$ and $A_n = \pi_d(Z_n)$, then M is closed and separable in X_d and $X_d \setminus M \subset \bigcup A_n$. Moreover, each A_n is clopen in X_d and $|\mu|_{\tilde{d},p}(A_n) \leq \epsilon$. Since M is separable, there exists an increasing sequence (δ_n) such that $M \subset \bigcup_{n=1}^\infty (X_d \setminus V_{\delta_n})$. Now $V_{\delta_n} \bigcap (X_d \setminus A_n) \downarrow \emptyset$ and hence there exists n with $|\mu|_{\tilde{d},p}(V_{\delta_n} \cap (X_d \setminus A_n)) \leq \epsilon$, which, together with the $|\mu|_{\tilde{d},p}(A_n) \leq \epsilon$, implies that $|\mu|_{\tilde{d},p}(V_{\delta_n}) \leq \epsilon$. This proves our claim.
Let now d, d_1 be continuous ultra-pseudometrics on X, with $d \leq d_1$, and let $f \in C_b(X, E)$ be d-continuous. The functions $h, h_1 : X_d \to E, h(\tilde{x}_d) = f(x) = h_1(\tilde{x})$ are well defined and continuous. Morerover $\int h d\mu_d = \int h_1 d\mu_{d_1}$. Indeed, let $\phi : X_{d_1} \to X_d, \tilde{x}_{d_1} \mapsto \tilde{x}_d$. Then ϕ is continuous and $\pi_d = \phi \circ \pi_{d_1}$. Let $S : C_b(X_d, E) \to C_b(X_{d_1}, E)$ be the induced linear map. Then S is $\beta_p - \beta_p$ continuous. Let $v_d : C_b(X_d, E) \to \mathbb{K}, v_d(g) = \int g d\mu_d$ and $v_{d_1} : C_b(X_{d_1}, E) \to \mathbb{K}, v_{d_1}(g) = \int g d\mu_{d_1}$. Then $S^* v_{d_1} = v_d$. Since $< S^* v_{d_1}, h >=< v_{d_1}, Sh >=< v_{d_1}, h_1 >$, we get that $v_d(h) = v_{d_1}(h_1)$.
Let d_o be an ultrametric on E generating its topology. If $f \in C_b(X, E)$, then we have a continuous ultra-pseudometric on X defined by $d(x, y) = d_o(f(x), f(y))$ and f is d-continuous.

If $h : X_d \to E, h(\tilde{x}_d) = f(x)$, then h is well defined continuous and $T_d h = f$. Now we define u_m on $C_b(X, E)$ as follows: For $f \in C_b(X, E)$, choose a continuous ultra-pseudometric d on X such that $f = T_d g$ for some $g \in C_b(X_d, E)$. Define $u_m(f) = \int g d\mu_d$. As we have shown above, u_m is well defined and linear.
Claim II: u_m is $\beta_{e,p}$-continuous. In fact, let $W = \{f \in C_b(X, E) : |u_m(f)| \leq 1\}$ and let d be a continuous ultra-pseudometric on X. Then $V = \{g \in C_b(X_d, E) : |\int g d\mu_d| \leq 1\}$ is a β_p-neighborhood of zero in $C_b(X_d, E)$ and $T_d(V) \subset W$, which proves our claim.
Now there exists $m_1 \in \mathcal{M}_{\sigma,p}(X, E')$ such that $u_m(f) = \int f dm_1$ for each $f \in C_b(X, E)$. It is easy to see that $m_1(A)s = m(A)s$ for each clopen subset A of X and each $s \in E$ and so $m = m_1$. It follows that every $f \in C_b(X, E)$ is m-integrable and $u_m(f) = \int f dm$ and so u_m is $\beta_{e,p}$-continuous. This completes the proof.

From the preceding Theorem we get

Theorem 6.9 *If E is metrizable, then $\mathcal{M}_s(X, E')$ is algebraically isomorphic with the dual space of $(C_b(X, E), \beta'_e)$ via the isomorphism $m \mapsto u_m$.*

Theorem 6.10 *Assume that E is metrizable and let $m \in \mathcal{M}_{\sigma,p}(X, E')$. Then:*
(1) If $(A_i)_{i \in I}$ is a clopen partition of X, then, for each $\epsilon > 0$, the set $I_\epsilon = \{i \in I : m_p(A_i) \geq \epsilon\}$ is finite. Moreover, $m \in \mathcal{M}_{u,p}(X, E')$.
(2) $\mathcal{M}_{\sigma,p}(X, E') = \mathcal{M}_{u,p}(X, E')$.

Proof: (1) let $\lambda \in \mathbb{K}, |\lambda| > 1$. For each i, there exist a clopen subset B_i of A_i and $s_i \in E, p(s_i \leq 1$, such that $|m(B_i)s_i| \geq (2|\lambda|)^{-1} m_p(A_i)$. For $J \subset I$ finite, set $g_J = \sum_{i \in J} \chi_{B_i} s_i$. If $g = \sum_{i \in I} \chi_{B_i} s_i$, then $g_J \to g$ pointwise. The family $\{g_J : J \text{ finite}\}$ is τ_p-bounded and p-equicontinuous. In view of Theorem 4.6, we have that $\int g dm = \lim_J \int g_J dm = \sum_i m(B_i)s_i$. Given now $\epsilon > 0$, there exists a finite subset J of I such that $|m(B_i)s_i| < \frac{\epsilon}{2|\lambda|}$, for $i \notin J$, and hence $I_\epsilon \subset J$.
Let now B be a clopen subset of $\bigcup_{i \notin I_\epsilon} A_i$. For each $s \in E$, we have $m(B)s = \sum_{i \notin I_\epsilon} m(A_i \cap B)s$. Since, for $i \notin I_\epsilon$ and $p(s) \leq 1$, we have $|m(A_i \cap B)s| \leq \epsilon p(s)$, it follows that $m_p(D) \leq \epsilon$. By Theorem 5.5, $m \in \mathcal{M}_{u,p}(X, E')$.
(2) If $\mu \in \mathcal{M}_{u,p}(X, E')$, then the map u_μ is $\beta_{u,p}$-continuous and hence $\beta_{e,p}$-continuous, which implies that $\mu \in \mathcal{M}_{s,p}(X, E')$.

Corollary 6.11 *If E is metrizable, then $\mathcal{M}_s(X, E') = \mathcal{M}_u(X, E')$ is the common dual space of $C_b(X, E)$ under the topologies β'_e and β'_u.*

Theorem 6.12 *Let E be metrizable and let H be a subset of $\mathcal{M}_s(X, E')$. Then H is β'_u-equicontinuous iff it is β'_e-equicontinuous.*

Proof: Assume that H β'_e-equicontinuous. Then there exists $p \in cs(E)$ such that H is $\beta_{e,p}$-equicontinuous. Since $\beta_{e,p} \leq \tau_p$, it follows that $\sup_{m \in H} m_p(X) > \infty$. Let now $G \in \Omega_u$ and let $(A_i)_{i \in I}$ be a clopen partition of X such that every $\bar{A}_i^{\beta_o X}$ is disjoint from G. For each i, there exist a clopen subset B_i of A_i and $s_i \in E, p(s_i) \leq 1$, such that $|m(B_i)s_i| \geq (2|\lambda|)^{-1} \sup_{m \in H} m_p(A_i)$ (where $|\lambda| > 1$). For $J \subset I$ finite, set $h_J = \sum_{i \in J} \chi_{B_i} s_i$. The net $(h_J)_J$ is τ_p-bounded and p-equicontinuous. Moreover $h_J \to h = \sum_{i \in I} \chi_{B_i} s_i$ pointwise. Let μ be a non-zero element of $\mathbb{K}$. There exists a finite subset J_o of I such that $h - h_J \in \mu H^o$ if $J_o \subset J$. It follows that $\sup_{m \in H} m_p(A_i) \leq 2|\mu\lambda|$ if $i \notin J_o$. If $D = \bigcup_{i \notin J_o} A_i$, then we get that $m_p(D) \leq 2|\mu\lambda|$ for all $m \in H$. If now $r > 0$, then there exists $J \subset I$ finite such that $m_p(D) \leq 1/r$, for all $m \in H$, where $D = \bigcup_{i \notin J} A_i$. The set $A = \bigcup_{i \in J} A_i$ is clopen and its closure in $\beta_o X$ is disjoint from G. Moreover $\{f \in C_b(X, E) : \|f\|_p \leq r, \|f\|_{A,p} \leq 1/\alpha\} \subset H^o$, where $\alpha > \sup_{m \in H} m_p(X)$. This proves that H^o is a $\beta_{G,p}$-neighborhood of zero and so it is β'_u-equicontinuous. Since β'_u is coarser than β'_e, the result follows.

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CONTROLLABILITY OF SECOND ORDER DELAY INTEGRODIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS

M. BENCHOHRA[1] **and S.K. NTOUYAS**[2]

[1] Department of Mathematics, University of Sidi Bel Abbes, BP 89, 22000 Sidi Bel Abbes, Algeria
e-mail: benchohra@yahoo.com
[2] Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
e-mail: sntouyas@cc.uoi.gr

Abstract

In this paper, we shall establish sufficient conditions for the controllability of second order delay integrodifferential inclusions in Banach spaces, with nonlocal conditions. By using suitable fixed point theorems we study the case when the multivalued map has convex as well nonconvex values.

Key words and phrases: Controllability, mild solution, contraction multivalued map, nonlocal condition, fixed point.
AMS (MOS) Subject Classifications: 93B05.

1 Introduction

In this paper, we shall establish sufficient conditions for the controllability of second order delay integrodifferential inclusions in Banach spaces, with nonlocal initial conditions. More precisely we consider the following semilinear system of the form

$$y'' - Ay \in \int_0^t K(t,s)F(s,y(\sigma(s)))ds + (Bu)(t), \quad t \in J := [0,b], \tag{1}$$

$$y(0) + f(y) = y_0, \; y'(0) = \eta, \tag{2}$$

where $F : J \times E \longrightarrow \mathcal{P}(E)$ is a multivalued map, $\sigma : J \to J$ is a continuous function such that $\sigma(t) \leq t, \forall t \in J$, $K : D \longrightarrow \mathrm{I\!R}$, $D = \{(t,s) \in J \times J : t \geq s\}$, $f : C(J,E) \longrightarrow E$ is a continuous given function, A is a linear infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathrm{I\!R}\}$ in a separable Banach space $E = (E, |\cdot|), y_0, \eta \in E$. Also the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. Finally B is a bounded linear operator from U to E.

The pioneering work on nonlocal evolution Cauchy problems is due to Byszewski. As pointed out by Byszewski [12], [11] the study of initial value problems with nonlocal conditions is of significance since they have applications in problems in physics and other areas of applied mathematics. In fact, more authors have paid attention to the research of initial value problems with nonlocal conditions, in the few past years. We refer to Balachandran and Chandrasekaran [2], Byszewski [11], [12], Ntouyas [26] and Ntouyas and Tsamatos [24], [25].

Initial value problems for second order semilinear equations with nonlocal conditions, were studied in Ntouyas and Tsamatos [25] and Ntouyas [26].

Recently, the authors in [4] studied the controllability of second order differential inclusions in Banach spaces with nonlocal conditions, in the case where the multivalued map has bounded, closed and convex values, by using the fixed point theorem of Martelli [23]. In this paper, we extend the results of [4] to second order delay integrodifferential inclusions in Banach spaces with nonlocal conditions, when the multivalued F has convex or nonconvex values. In the first case a fixed point theorem due to Martelli is used and in the later a fixed point theorem for contraction multivalued maps, due to Covitz and Nadler [15].

For other recent controllability results for first and second order differential and integrodifferential inclusions in Banach spaces with nonlocal conditions see [5]-[10].

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$C(J, E)$ is the Banach space of continuous functions from J into E normed by

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E.

A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [29]).

$L^1(J, E)$ denotes the Banach space of measurable functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt \quad \text{for all} \quad y \in L^1(J, E).$$

Let $(X, |\cdot|)$ be a Banach space. A multivalued map $G : X \longrightarrow \mathcal{P}(E)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for any bounded set B of X, that is $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$. G is called *upper semicontinuous (u.s.c.)* on X if, for each $x_0 \in X$, the set $G(x_0)$ is a nonempty, closed subset of X, and if, for each open set V of X containing $G(x_0)$, there exists an open neighbourhood U of x_0 such that $G(U) \subseteq V$.

G is said to be *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_0,\ y_n \longrightarrow y_0,\ y_n \in G(x_n)$ imply $y_0 \in G(x_0)$). G has a *fixed point* if there is $x \in X$ such that $x \in G(x)$.

$P(X) = \{Y \in \mathcal{P}(X) : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$, and $P_c(X) = \{Y \in P(X) : Y \text{ convex}\}$. A multivalued map $G : J \longrightarrow P_{cl}(X)$ is said to be *measurable* if for each $x \in X$ the function $Y : J \longrightarrow \mathbb{R}_+$, defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\},$$

is measurable. For more details on multivalued maps we refer to the books of Deimling [16], Gorniewicz [19] and Hu and Papageorgiou [21].

An upper semi-continuous map $G : X \longrightarrow \mathcal{P}(X)$ is said to be *condensing* if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompacteness. For properties of the Kuratowski measure, we refer to Banas and Goebel [3]. We remark that a completely continuous multivalued map is the easiest example of a condensing map.

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if
(i) $C(0) = I$ (I is the identity operator in E),
(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$,
(iii) the map $t \longmapsto C(t)y$ is strongly continuous for each $y \in E$;

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y\,ds, \quad y \in E,\ t \in \mathbb{R}.$$

The infinitesimal generator $A : E \longrightarrow E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2}C(t)y\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the book of Goldstein [18], Heikkila and Lakshmikantham [20], Fattorini [17] and to the papers of Travis and Webb [27], [28].

3 Existence result: The convex case

Assume in this section that $F : J \times E \to \mathcal{P}(E)$ is a bounded, closed and convex valued multivalued map.

We will need the following assumptions:

(H1) A is the infinitesimal generator of a given strongly continuous and bounded cosine family $\{C(t) : t \in J\}$ with $M = \sup\{|C(t)|; t \in J\}$;

(H2) $F : J \times E \longrightarrow P_{b,cl,c}(E); (t, y) \longmapsto F(t, y)$ is measurable with respect to t for each $y \in E$, u.s.c. with respect to y for each $t \in J$ and for each fixed $y \in C(J, E)$ the set

$$S_{F,y} = \Big\{g \in L^1(J, E) : \ g(t) \in F(t, y(\sigma(t))) \text{ for a.e. } t \in J\Big\}$$

is nonempty;

(H3) there exists a constant L such that

$$|f(y)| \leq L, \quad \text{for each} \ \ y \in C(J, E);$$

(H4) for each $t \in J$, $K(t, s)$ is measurable on $[0, t]$ and

$$K(t) = \text{ess sup}\{|K(t, s)|, \quad 0 \leq s \leq t\},$$

is bounded on J;

(H5) the map $t \longmapsto K_t$ is continuous from J to $L^\infty(J, \mathbb{R})$; here $K_t(s) = K(t, s)$;

(H6) $\sigma : J \to J$ is a continuous function, such that $\sigma(t) \leq t, \forall t \in J$.

(H7) The linear operator $W : L^2(J, U) \to E$, defined by

$$Wu = \int_0^b S(b - s)Bu(s)\, ds,$$

has an invertible operator $\widetilde{W}^{-1}$ which takes values in $L^2(J, U)/kerW$ and there exist positive constants M_1 and M_2 such that $|B| \leq M_1$ and $|W^{-1}| \leq M_2$.

(H8) $\|F(t, y)\| := \sup\{|v| : v \in F(t, y)\} \leq p(t)\psi(|y|)$ for almost all $t \in J$ and all $y \in E$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow (0, \infty)$ is continuous and increasing with

$$Mb \sup_{t \in J} K(t) \int_0^b p(s) ds < \int_c^\infty \frac{du}{\psi(u)};$$

where $c = M(|y_0| + L + Mb|\eta| + M_0)$ and

$$\begin{aligned} M_0 \ &= bM_1M_2\Big[|x_1| + L + M|y_0| + ML + bM|\eta| \\ &+bM \sup_{t \in J} K(t) \int_0^b p(s)\psi(|y(s)|)ds\Big]. \end{aligned}$$

(H9) For each bounded set $D \subset C(J,E)$, and $t \in J$ the set

$$\left\{C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)\int_0^s K(s,u)g(u)duds : g \in S_{F,D}\right\}$$

is relatively compact in E, where $S_{F,D} = \cup\{S_{F,y} : y \in D\}$.

Remark 3.1 *(i) If $dimE < \infty$, then for each $y \in C(J,E)$ the set $S_{F,y}$ is nonempty (see Lasota and Opial [22]).*
(ii) If $dimE = \infty$ and $y \in C(J,E)$ the set $S_{F,y}$ is nonempty if and only if the function $Y : J \longrightarrow \mathbb{R}$ defined by

$$Y(t) := \inf\{|v| : v \in F(t,y)\}$$

belongs to $L^1(J,\mathbb{R})$ (see Hu and Papageorgiou [21]).
(iii) Examples with $W : L^2(J,U) \to E$ such that W^{-1} exists and is bounded are discussed in [13].
(iv) If we assume that $C(t)$, $t > 0$ is completely continuous then (H9) is satisfied.

Definition 3.1 *A function $y \in C(J,E)$ is said to be a mild solution of (1)-(2) on J if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y(\sigma(t)))$ a.e. on J, $y(0) + f(y) = y_0$, and*

$$\begin{aligned} y(t) &= C(t)y_0 - C(t)f(y) + S(t)\eta + \int_0^t S(t-s)(Bu)(s)\,ds \\ &\quad + \int_0^t S(t-s)\int_0^s K(s,\tau)v(\tau)d\tau ds. \end{aligned}$$

Definition 3.2 *The system (1)-(2) is said to be nonlocally controllable on the interval J, if for every $y_0, \eta, x_1 \in E$, there exists a control $u \in L^2(J,U)$, such that the mild solution $y(t)$ of (1)-(2) satisfies $y(b) + f(y) = x_1$.*

The following lemmas are crucial in the proof of our main theorem.

Lemma 3.1 *[22] Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying (H2) and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$, then the operator*

$$\Gamma \circ S_F : C(I,X) \longrightarrow P_{b,cl,c}(C(I,X)),\ y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 3.2 *[23]. Let X be a Banach space and $N : X \longrightarrow P_{b,cl,c}(X)$ an u.s.c. condensing map. If the set*

$$\Omega := \{y \in X : \lambda y \in N(y) \ \textit{for some}\ \lambda > 1\}$$

is bounded, then N has a fixed point.

Theorem 3.1 *Let $f : C(J,E) \longrightarrow E$ be a continuous function. Assume that hypotheses (H1)-(H9) are satisfied. Then the problem (1)-(2) is nonlocally controllable on J.*

Proof. Using hypothesis (H7) for an arbirtary function $y(\cdot)$ define the control

$$\begin{aligned} u_y(t) &= \widetilde{W}^{-1}\Big[x_1 - f(y) - C(b)y_0 + C(b)f(y) - S(b)\eta \\ &\quad - \int_0^b S(b-s)\int_0^s K(s,\tau)g(\tau)\,d\tau ds\Big](t), \end{aligned}$$

where

$$g \in S_{F,y} = \Big\{g \in L^1(J,E) : g(t) \in F(t, y(\sigma(t))) \text{ for a.e. } t \in J\Big\}.$$

We shall now show that when using this control, the operator $N : C(J,E) \longrightarrow \mathcal{P}(C(J,E))$ defined by

$$\begin{aligned} N(y) := \Big\{h \in C(J,E) : h(t) &= C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)\,d\tau ds : g \in S_{F,y}\Big\} \end{aligned}$$

has a fixed point. This fixed point is then a solution of the system (1)-(2).

Clearly $x_1 - f(y) \in N(y)(b)$.

We shall show that N satisfies the assumtions of Lemma 3.2. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in C(J,E)$.*

Indeed, if $h_1,\ h_2$ belong to $N(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$\begin{aligned} h_i(t) &= C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s K(s,\tau)g_i(\tau)d\tau ds, \quad i = 1,2. \end{aligned}$$

Let $0 \le \alpha \le 1$. Then for each $t \in J$ we have

$$\begin{aligned} (\alpha h_1 + (1-\alpha)h_2)(t) &= C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s K(s,\tau)[\alpha g_1(\tau) + (1-\alpha)g_2(\tau)]d\tau ds. \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values) then

$$\alpha h_1 + (1-\alpha)h_2 \in N(y).$$

Step 2: *N is bounded on bounded sets of $C(J, E)$.*

Indeed, it is enough to show that there exists a positive constant l such that for each $h \in N(y), y \in B_r = \{y \in C(J, E) : \|y\|_\infty \leq r\}$ one has $\|h\|_\infty \leq l$. If $h \in N(y)$, then there exists $g \in S_{F,y}$ such that

$$\begin{aligned} h(t) &= C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)\, ds \\ &\quad + \int_0^t S(t-s) \int_0^s K(s,\tau)g(\tau)d\tau ds, \quad t \in J. \end{aligned}$$

By (H3)-(H7) we have for each $t \in J$ that

$$\begin{aligned} |h(t)| &\leq |C(t)||y_0| + |C(t)||f(y)| + |S(t)||\eta| + \left\| \int_0^t S(t-s)(Bu_y)(s)\, ds \right\| \\ &\quad + \left\| \int_0^t S(t-s) \int_0^s K(s,\tau)g(\tau)d\tau ds \right\| \\ &\leq M|y_0| + ML + Mb|\eta| \\ &\quad + bMM_1M_2[|x_1| + L + M|y_0| + ML + bM|\eta| \\ &\quad + Mb \sup_{t\in J} K(t)\|p\|_{L^1} \sup_{t\in J} \psi(|y(t)|)] \\ &\quad + M \int_0^t \int_0^s |K(s,u)|p(u)\psi(|y(\sigma(u))|)duds \\ &\leq M|y_0| + ML + b|\eta| \\ &\quad + bMM_1M_2[|x_1| + L + M|y_0| + ML + bM|\eta| \\ &\quad + Mb \sup_{t\in J} K(t)\|p\|_{L^1} \sup_{t\in J} \psi(|y(t)|)] \\ &\quad + Mb \sup_{t\in J} K(t)|p|_{L^1} \sup_{t\in J} \psi(|y(t)|). \end{aligned}$$

Then for each $h \in N(y)$ we have

$$\begin{aligned} \|h\|_\infty &\leq M|y_0| + ML + b|\eta| + MM_0 + Mb \sup_{t\in J} K(t)\|p\|_{L^1} \sup_{t\in J} \psi(|y(t)|)] \\ &\quad + Mb \sup_{t\in J} K(t)|p|_{L^1} \sup_{t\in J} \psi(|y(t)|) := l. \end{aligned}$$

Step 3: *N sends bounded sets of $C(J, E)$ into equicontinuous sets.*

Let $t_1, t_2 \in J, t_1 < t_2$ and B_r be a bounded set of $C(J, E)$. For each $y \in B_r$ and $h \in N(y)$, there exists $g \in S_{F,y}$ such that

$$h(t) = C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)ds$$

$$+\int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)d\tau ds, \quad t\in J.$$

Thus

$$\begin{aligned}
|h(t_2)-h(t_1)| \leq\; & |(C(t_2)-C(t_1))y_0| + L|C(t_2)-C(t_1)| + |S(t_2)-S(t_1)||\eta| \\
& + \Big\| \int_0^{t_2} [S(t_2-s)-S(t_1-s)]\, BW^{-1}\Big[x_1 - f(y) - C(b)y_0 \\
& + C(b)f(y) - S(b)\eta - \int_0^b S(b-s)\int_0^s K(s,\tau)g(\tau)d\tau ds\Big](\eta)d\eta \Big\| \\
& + \Big\| \int_{t_1}^{t_2} S(t_1-s)BW^{-1}\Big[x_1 - f(y) - C(b)y_0 + C(b)f(y) - S(b)\eta \\
& + \int_0^b S(b-s)\int_0^s K(s,\tau)g(\tau)d\tau ds\Big](\eta)d\eta \Big\| \\
& + \Big\| \int_0^{t_2} [S(t_2-s)-S(t_1-s)] \int_0^s K(s,\tau)g(\tau)d\tau ds \Big\| \\
& + \Big\| \int_{t_1}^{t_2} S(t_1-s)\int_0^s K(s,\tau)g(\tau)d\tau ds \Big\| \\
\leq\; & |C(t_2)-C(t_1)||y_0| + L|C(t_2)-C(t_1)| + |S(t_2)-S(t_1)||\eta| \\
& + \int_0^{t_2} |S(t_2-s)-S(t_1-s)|\, M_1M_2\Big[|x_1| + L + M|y_0| \\
& + ML + bM|\eta| + Mb\sup_{t\in J} K(t)\int_0^b p(s)\psi(|y(s)|)ds\Big](\eta)d\eta \\
& + \int_{t_1}^{t_2} |S(t_1-s)| M_1M_2\Big[|x_1| + L + M|y_0| + ML + bM|\eta| \\
& + Mb\sup_{t\in J} K(t)\int_0^b p(s)\psi(|y(s)|)ds\Big](\eta)d\eta \\
& + \sup_{t\in J} K(t)\Big\| \int_0^{t_2} [S(t_2-s)-S(t_1-s)] \int_0^s g(\tau)d\tau ds \Big\| \\
& + M\sup_{t\in J} K(t)(t_2-t_1)\int_0^b \|g(s)\|ds.
\end{aligned}$$

As $t_2 \longrightarrow t_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Step 2, Step 3 and (H9), together with the Arzela-Ascoli theorem, we can conclude that N is completely continuous, and therefore, a condensing multivalued map.

Step 4: *N has a closed graph.*

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) &= C(t)y_0 - C(t)f(y_n) + S(t)\eta + \int_0^t S(t-s)(Bu_{y_n})(s)ds \\ &\quad + \int_0^t S(t-s)\int_0^s K(s,\tau)g_n(\tau)d\tau ds, \quad t \in J, \end{aligned}$$

where

$$\begin{aligned} u_{y_n}(t) &= \widetilde{W}^{-1}\Big[\eta - f(y_n) - C(b)y_0 + C(b)f(y_n) - S(b)\eta \\ &\quad - \int_0^b S(b-s)\int_0^s K(s,\tau)g_n(\tau)\,d\tau ds\Big](t). \end{aligned}$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that

$$\begin{aligned} h_*(t) &= C(t)y_0 - C(t)f(y_*) + S(t)\eta + \int_0^t S(t-s)(Bu_{y_*})(s)ds \\ &\quad + \int_0^t S(t-s)\int_0^s K(s,\tau)g_*(\tau)d\tau ds, \quad t \in J, \end{aligned}$$

where

$$\begin{aligned} u_{y_*}(t) &= \widetilde{W}^{-1}\Big[\eta - f(y_*) - C(b)y_0 + C(b)f(y_*) - S(b)\eta \\ &\quad - \int_0^b S(b-s)\int_0^s K(s,\tau)g_*(\tau)\,d\tau ds\Big](t). \end{aligned}$$

Set

$$\overline{u}_y(t) = \widetilde{W}^{-1}\Big[\eta - f(y) - C(b)y_0 + C(b)f(y) - S(b)\eta\Big].$$

Since f, W^{-1} are continuous, then $\overline{u}_{y_n}(t) \longrightarrow \overline{u}_{y_*}(t)$ for $t \in J$.
Clearly we have that

$$\begin{aligned} &\Big\|\Big(h_n - C(t)y_0 + C(t)f(y_n) - S(t)\eta - \int_0^t S(t-s)(B\overline{u}_{y_n})(s)ds\Big) \\ &- \Big(h_* - C(t)y_0 + C(t)f(y_*) - S(t)\eta - \int_0^t S(t-s)(B\overline{u}_{y_*})(s)ds\Big)\Big\|_\infty \to 0, \end{aligned}$$

as $n \longrightarrow \infty$.

Consider the operator

$$\Gamma : L^1(J,E) \longrightarrow C(J,E)$$

$$g \longmapsto \Gamma(g)(t) = \int_0^t S(t-s)\Big[BW^{-1}\Big(\int_0^b S(b-s)\int_0^s K(s,\tau)g(\tau)d\tau ds\Big)(s)\Big]ds + \int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)d\tau ds.$$

Clearly, Γ is linear and continuous. Indeed one has

$$\|\Gamma g\|_\infty \leq b^2 M \sup_{t\in J} K(t)(bMM_1M_2+1)\|g\|_{L^1}.$$

¿From Lemma 3.2, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have that

$$h_n(t) - C(t)y_0 + C(t)f(y_n) - S(t)\eta - \int_0^t S(t-s)(B\overline{u}_{y_n})(s)ds \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows from Lemma 3.2 that

$$\begin{aligned} & h_*(t) - C(t)y_0 + C(t)f(y_*) - S(t)\eta - \int_0^t S(t-s)(B\overline{u}_{y_*})(s)ds \\ = & \int_0^t S(t-s)\Big[BW^{-1}\Big(\int_0^b S(b-s)\int_0^s K(s,\tau)g_*(\tau)d\tau ds\Big)(s)\Big]ds \\ + & \int_0^t S(t-s)\int_0^s K(s,\tau)g_*(\tau)d\tau ds \end{aligned}$$

for some $g_* \in S_{F,y_*}$.

Step 5: *The set*

$$\Omega := \{y \in C(J,E) : \lambda y \in N(y), \quad \textit{for some} \quad \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$\begin{aligned} y(t) = {} & \lambda^{-1}C(t)y_0 - \lambda^{-1}C(t)f(y) + \lambda^{-1}S(t)\eta \\ & +\lambda^{-1}\int_0^t S(t-s)BW^{-1}\Big[x_1 - f(y) - C(b)y_0 + C(b)f(y) + S(t)\eta \\ & - \int_0^b S(b-s)\int_0^s K(s,\tau)g(\tau)\,d\tau ds\Big](\eta)d\eta \\ & +\lambda^{-1}\int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)d\tau ds, \ t \in J. \end{aligned}$$

This implies by (H3)-(H8) that for each $t \in J$ we have

$$|y(t)| \leq M|y_0| + ML + bM|\eta|$$

$$+bMM_1M_2\Big[|x_1| + L + M|y_0| + ML + bM|\eta|$$

$$+bM\sup_{t\in J} K(t)\int_0^b p(s)\psi(|y(\sigma(s))|)ds\Big]$$

$$+M\left|\int_0^t\int_0^s K(s,\tau)g(\tau)d\tau ds\right|$$

$$\leq M|y_0| + ML + bM|\eta|$$

$$+bMM_1M_2\Big[|x_1| + L + M|y_0| + ML + bM|\eta|$$

$$+bM\sup_{t\in J} K(t)\int_0^b p(s)\psi(|y(\sigma(s))|)ds\Big]$$

$$+Mb\sup_{t\in J} K(t)\int_0^t p(s)\psi(|y(\sigma(s))|)ds.$$

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$v(0) = M|y_0| + ML + MM_0, \quad |y(t)| \leq v(t), \; t \in J,$$

and

$$v'(t) = Mb\sup_{t\in J} K(t)p(t)\psi(|y(\sigma(t))|), \; t \in J.$$

Using the nondecreasing character of ψ and the fact that $\sigma(t) \leq t, \forall t \in J$ we get

$$v'(t) \leq Mb\sup_{t\in J} K(t)p(t)\psi(v(t)), \; t \in J.$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq Mb\sup_{t\in J} K(t)\int_0^b p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

This inequality implies that there exists a constant d such that $v(t) \leq d, \; t \in J$, and hence $\|y\|_\infty \leq d$ where d depends only on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J, E)$. As a consequence of Lemma 3.2 we deduce that N has a fixed point and thus the system (1)-(2) is nonlocally controllable on J. ■

4 Existence Result: The nonconvex case

In this section we consider the problems (1)-(2), with a nonconvex valued right hand side.

Let (X,d) be the metric space induced from the normed space $(X,|\cdot|)$.
Consider $H_d : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b),\ \ d(a,B) = \inf_{b\in B} d(a,b)$.
Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space.

Definition 4.1 *A multivalued operator $N : X \to P_{cl}(X)$ is called*

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x,y), \quad \textit{for each } x,\ y \in X,$$

b) contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

c) N has a fixed point *if there is $x \in X$ such that $x \in N(x)$. The fixed point set of the multivalued operator N will be denoted by $FixN$.*

Our considerations are based on the following fixed point theorem for contraction multivalued operators given by Covitz and Nadler in 1970 [15] (see also Deimling, [16] Theorem 11.1).

Lemma 4.1 *Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

We will need the following assumptions:

(A1) $F : [0,b] \times E \longrightarrow P_{cl}(E)$ is integrably bounded and has the property that $F(\cdot, y) : [0,b] \to P_{cl}(E)$ is measurable for each $y \in E$.

(A2) $H_d(F(t,y), F(t,\overline{y})) \leq l(t)\|y-\overline{y}\|$, for almost each $t \in [0,b]$ and $y, \overline{y} \in E$, where $l \in L^1([0,b], \mathbb{R})$.

(A3) $\|f(y) - f(\overline{y})\| \leq c\|y - \overline{y}\|$, for each $t \in [0,b]$ and $y, \overline{y} \in C([0,b], E)$, where c is a nonnegative constant.

Now we are able to state and prove our main result for this section.

Theorem 4.1 *Assume that hypotheses (H1), (H4)-(H7) and (A1)-(A3) are satisfied. Then the problem (1)-(2) is nonlocally controllable on J, provided*

$$Mc + MM_1M_2bc + M^2M_1M_2bc + \frac{M^2M_1M_2b^2\sup_{t\in J}K(t)}{\tau} + \frac{M}{\tau} < 1.$$

Proof. Using hypothesis (H7) for an arbirtary function $y(\cdot)$ define the control

$$\begin{aligned} u_y(t) &= \widetilde{W}^{-1}\Big[x_1 - f(y) - C(b)y_0 + C(b)f(y) - S(b)\eta \\ &\quad - \int_0^b S(b-s)\int_0^s K(s,\tau)g(\tau)\,d\tau ds\Big](t), \end{aligned}$$

where $g \in S_{F,y}$.

Remark 4.1 *For each $y \in C([0,b],E)$, the set $S_{F,y}$ is nonempty, since by (A1), F has a measurable selection (see [14], Theorem III.6).*

We shall now show that, when using this control, the operator $N : C(J,E) \longrightarrow \mathcal{P}(C(J,E))$ defined by

$$\begin{aligned} N(y) := \Big\{h \in C(J,E) : h(t) &= C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)d\tau ds : g \in S_{F,y}\Big\} \end{aligned}$$

has a fixed point. This fixed point is then a solution of the system (1)-(2).

Clearly $x_1 - f(y) \in N(y)(b)$.

We shall show that N satisfies the assumptions of Lemma 4.1. The proof will be given in two steps.

Step 1: $N(y) \in P_{cl}(C([0,b],E))$ *for each* $y \in C([0,b],E)$.

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \longrightarrow \tilde{y}$ in $C([0,b],E)$. Then $\tilde{y} \in C([0,b],E)$ and

$$\begin{aligned} y_n(t) &\in C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s K(s,\tau)F(\tau, y(\sigma(\tau)))d\tau ds, \ t \in J. \end{aligned}$$

Using (A1) one can easily show by standard argument that

$$\int_0^t S(t-s)\int_0^s K(s,\tau)F(\tau, y(\sigma(\tau)))d\tau ds$$

is closed for each $t \in [0,b]$. Then

$$\begin{aligned} y_n(t) \longrightarrow \tilde{y}(t) &\in C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s K(s,\tau)F(\tau, y(\sigma(\tau)))d\tau ds, \ t \in J. \end{aligned}$$

So $\tilde{y} \in N(y)$.

Step 2: $H_d(N(\eta), N(y_2)) \leq \gamma\|y_1 - y_2\|$ *for each* $y_1, y_2 \in C([0,b],E)$ *(where* $\gamma < 1$*).*

Let $y_1, y_2 \in C([0,b],E)$ and $h_1 \in N(y_1)$. Then there exists $g_1(t) \in F(t, y_1(\sigma(t)))$ such that

$$\begin{aligned} h_1(t) &= C(t)(y_0 - f(y_1)) + S(t)\eta + \int_0^t S(t-s)(Bu_{y_1})(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s g_1(\tau)d\tau ds, \; t \in J. \end{aligned}$$

From (H3) it follows that

$$\begin{aligned} H_d(F(t,y_1(\sigma(t))), F(t,y_2(\sigma(t)))) &\le l(t)\|y_1(\sigma(t)) - y_2(\sigma(t))\| \\ &\le l(t)\|y_1(t) - y_2(t)\|, \; t \in J. \end{aligned}$$

Hence there is $w \in F(t, y_2(\sigma(t)))$ such that

$$\|g_1(t) - w\| \le l(t)\|y_1(t) - y_2(t)\|, \quad t \in J.$$

Consider $U : [0,b] \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : \|g_1(t) - w\| \le l(t)\|y_1(t) - y_2(t)\|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, y_2(\sigma(t)))$ is measurable (see Proposition III.4 in [14]) there exists $g_2(t)$ a measurable selection for V. So, $g_2(t) \in F(t, y_2(\sigma(t)))$ and

$$\|g_1(t) - g_2(t)\| \le l(t)\|y_1(t) - y_2(t)\|, \quad \text{for each } \; t \in J.$$

Let us define for each $t \in J$

$$\begin{aligned} h_2(t) &= C(t)(y_0 - f(y_2)) + S(t)\eta + \int_0^t S(t-s)(Bu_{y_2})(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s g_2(\tau)d\tau ds. \end{aligned}$$

Then we have

$$\begin{aligned} \|h_1(t) - h_2(t)\| &\le M\|f(y_1) - f(y_2)\| + MM_1\int_0^t \|u_{y_1}(s) - u_{y_2}(s)\|ds \\ &+ Mb\int_0^t \|g_1(s) - g_2(s)\|\,ds \\ &\le Mc\|y_1 - y_2\| + MM_1M_2\int_0^t [\|f(y_1) - f(y_2)\| \\ &+ M\|f(y_1) - f(y_2)\| + b^2 \sup_{t\in J} K(t)M\|g_1(s) - g_2(s)\|]ds \\ &+ Mb\int_0^t \|g_1(s) - g_2(s)\|ds \end{aligned}$$

$$
\begin{aligned}
&\leq Mc\|y_1 - y_2\| + MM_1M_2[bc\|y_1 - y_2\| + bMc\|y_1 - y_2\| \\
&\quad +b^2 \sup_{t\in J} M \int_0^t l(s)\|y_1(s) - y_2(s)\|ds] \\
&\quad +M \int_0^t l(s)\|y_1(s) - y_2(s)\|ds \\
&= Mce^{\tau L(t)}\|y_1 - y_2\|_B + MM_1M_2bce^{\tau L(t)}\|y_1 - y_2\|_B \\
&\quad +M^2M_1M_2bce^{\tau L(t)}\|y_1 - y_2\|_B \\
&\quad +M^2M_1M_2b^2 \sup_{t\in J} K(t) \int_0^t l(s)e^{-\tau L(s)}e^{\tau L(s)}\|y_1(s) - y_2(s)\|ds \\
&\quad +M \int_0^t l(s)e^{-\tau L(s)}e^{\tau L(s)}\|y_1(s) - y_2(s)\|ds \\
&\leq Mce^{\tau L(t)}\|y_1 - y_2\|_B + MM_1M_2bce^{\tau L(t)}\|y_1 - y_2\|_B \\
&\quad +M^2M_1M_2bce^{\tau L(t)}\|y_1 - y_2\|_B \\
&\quad +M^2M_1M_2b^2 \sup_{t\in J} K(t)\|y_1 - y_2\|_B\frac{1}{\tau}e^{\tau L(t)} \\
&\quad +M\|y_1 - y_2\|_B\frac{1}{\tau}e^{\tau L(t)},
\end{aligned}
$$

where $L(t) = \int_0^t l(s)ds$, and $\|\cdot\|_B$ is the Bielecki-type norm on $C([0,b],E)$ defined by

$$\|y\|_B = \max_{t\in[0,b]}\{\|y(t)\|e^{-\tau L(t)}\}.$$

Then

$$\|h_1-h_2\|_B \leq \left[Mc + MM_1M_2bc + M^2M_1M_2bc + \frac{M^2M_1M_2b^2 \sup_{t\in J} K(t)}{\tau} + \frac{M}{\tau}\right] \|y_1-y_2\|_B.$$

By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$
\begin{aligned}
&H_d(N(y_1), N(y_2)) \leq \\
&\left[Mc + MM_1M_2bc + M^2M_1M_2bc + \frac{M^2M_1M_2b^2 \sup_{t\in J} K(t)}{\tau} + \frac{M}{\tau}\right] \|y_1 - y_2\|_B.
\end{aligned}
$$

Then N is a contraction and thus, by Lemma 4.1, it has a fixed point y, which is a mild solution to (1)-(2). ■

5 An Example

Consider the following partial integrodifferential equation of the form

$$\begin{aligned} z_{tt}(t,y) - z_{yy}(t,y) &= \int_0^t K(t,s)q(s,z(s,y(s-\tau)))\,ds \\ &\quad +Bu(t),\ \ 0 \le y \le \pi,\ t \in J \end{aligned} \tag{3}$$

$$\begin{aligned} &z(t,0) = z(t,\pi) = 0, \\ &z(0,y) + z(1,y) = z_0(y), \\ &z_t(0,y) = z_1(y) \end{aligned} \tag{4}$$

where $q : J \times E \to E$, is continuous.

Let $E = L^2[0,\pi]$ and define $A : E \to E$ by $Aw = w''$ with domain

$$D(A) = \{w \in E, w, w' \text{ are absolutely continuous, } w'' \in E, w(0) = w(\pi) = 0\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} n^2(w, w_n)w_n,\ \ w \in D(A)$$

where $w_n(s) = \sqrt{\frac{2}{\pi}}\, sin\, ns$, $n = 1, 2, \ldots$ is the orthogonal set of eigenvectors in A. It is easily shown that A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in \mathbb{R}$ in E given by

$$C(t)w = \sum_{n=1}^{\infty} cosnt(w, w_n)w_n, \quad w \in E$$

and that the associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{1}{n} sinnt(w, w_n)w_n, \quad w \in E.$$

Assume that the operator $B : U \to Y, U \subset J$, is a bounded linear operator and the operator

$$Wu = \int_0^b S(b-s)Bu(s)ds$$

has a bounded invertible operator $\widetilde{W}^{-1}$ which takes values in $L^2(J,U)\backslash kerW$.

Assume that there exists an integrable function $p : J \to [0,\infty)$ such that

$$|q(t, w(t))| \le p(t)\psi(|w|)$$

where $\psi : [0,\infty) \to (0,\infty)$ is continuous and nondecreasing with

$$Mb \sup_{t\in J} K(t) \int_0^b p(t)dt < \int_c^{\infty} \frac{ds}{\psi(s)}$$

where c is a constant.

Then the problem (1)-(2) is an abstract formulation of (3)-(4). Since all the conditions of Theorem 4.1 are satisfied, the problem (3)-(4) is nonlocally controllable on J.

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OSCILLATION CRITERIA FOR FIRST-ORDER DELAY EQUATIONS

by
Y.G. Sficas and I.P. Stavroulakis
Department of Mathematics
University of Ioannina
451 10 Ioannina,Greece
ipstav@cc.uoi.gr

ABSTRACT

This paper is concerned with the oscillatory behavior of first-order delay differential equations of the form

$$x'(t) + p(t)x(\tau(t)) = 0, \qquad t \geq t_0, \tag{1}$$

where $p, \tau \in C([t_0, \infty), \mathbb{R}^+), \mathbb{R}^+ = [0, \infty), \tau(t)$ is non-decreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$. Let the numbers k and L be defined by

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \text{ and } L = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds.$$

It is proved here that when $L < 1$ and $0 < k \leq \frac{1}{e}$ all solutions of Eq. (1) oscillate in several cases in which the condition

$$L > \frac{\ln \lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2k\lambda_1}}{\lambda_1}$$

holds, where λ_1 is the smaller root of the equation $\lambda = e^{k\lambda}$.

[0] *Key Words:* Oscillation; delay differential equations.
1991 *Mathematics Subject Classification:* Primary 34K15; Secondary 34C10

1 Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions of the differential equation

$$x'(t) + p(t)\,x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

where the functions $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$(here $\mathbb{R}^+ = [0, \infty)$), $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$, has been the subject of many investigations. See, for example, [1-27] and the references cited therein.

By a solution of Eq. (1) we understand a continuously differentiable function defined on $[\tau(T_0), \infty)$ for some $T_0 \geq t_0$ and such that (1) is satisfied for $t \geq T_0$. Such a solution is called oscillatory if it has arbitrarily large zeros, and otherwise it is called nonoscillatory.

The first systematic study for the oscillation of all solutions of Eq. (1) was made by Myshkis. In 1950 [24] he proved that every solution of Eq. (1) oscillates if

$$\limsup_{t\to\infty}[t - \tau(t)] < \infty, \quad \liminf_{t\to\infty}[t - \tau(t)] \cdot \liminf_{t\to\infty} p(t) > \frac{1}{e}. \tag{C_1}$$

In 1972, Ladas, Lakshmikantham and Papadakis [19] proved that the same conclusion holds if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1. \tag{C_2}$$

In 1979 Ladas [18] and in 1982 Koplatadze and Chanturija [14] improved (C_1) to

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}. \tag{C_3}$$

Concerning the constant $\frac{1}{e}$ in (C_3), it is to be pointed out that if the inequality

$$\int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e}$$

holds eventually, then, according to a result in [14], (1) has a non-oscillatory solution.

In 1982 Ladas, Sficas and Stavroulakis [20] and in 1984 Fukagai and Kusano [10] established oscillation criteria (of the type of conditions $(C_2)$and (C_3)) for Eq. (1) with *oscillating* coefficient $p(t)$.

It is obvious that there is a gap between the conditions (C_2) and (C_3) when the limit

$$\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds$$

does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors.

In 1988, Erbe and Zhang [9] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible nonoscillatory solutions $x(t)$ of Eq. (1). Their result, when formulated in terms of the numbers k and L defined by

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \quad \text{and} \quad L = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds,$$

says that all the solutions of Eq (1) are oscillatory, if $0 < k \leq \frac{1}{e}$ and

$$L > 1 - \frac{k^2}{4}. \tag{C_4}$$

Since then several authors tried to obtain better results by improving the upper bound for $x(\tau(t))/x(t)$. In 1991 Jian Chao [2] derived the condition

$$L > 1 - \frac{k^2}{2(1-k)}, \tag{C_5}$$

while in 1992 Yu and Wang [26] and Yu, Wang, Zhang and Qian [27] obtained the condition

$$L > 1 - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2}. \tag{C_6}$$

In 1990 Elbert and Stavroulakis [7] and in 1991 Kwong [17], using different techniques, improved (C_4), in the case where $0 < k \leq \frac{1}{e}$, to the conditions

$$L > 1 - (1 - \frac{1}{\sqrt{\lambda_1}})^2 \tag{C_7}$$

and

$$L > \frac{\ln \lambda_1 + 1}{\lambda_1}, \tag{C_8}$$

respectively, where $\lambda_1$is the smaller root of the equation

$$\lambda = e^{\kappa\lambda}. \tag{2}$$

In 1994 Koplatadze and Kvinikadze [14] improved (C_6), while in 1998 Philos and Sficas [25] , in 1999 Jaroš and Stavroulakis [11] and in 2000 Kon, Sficas and Stavroulakis [12] derived the conditions

$$L > 1 - \frac{k^2}{2(1-k)} - \frac{k^2}{2}\lambda_1, \tag{C_9}$$

$$L > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2} \tag{C_{10}}$$

and

$$L > 2k + \frac{2}{\lambda_1} - 1, \tag{C_{11}}$$

respectively, where λ_1 is the smaller root of Eq. (2).

Following this historical (and chronological) review we also mention that in the case where

$$\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \frac{1}{e}$$

this problem has been studied in 1993 by Elbert and Stavroulakis [8] and in 1995 by Kozakiewicz [16], Li [22], [23] and by Domshlak and Stavroulakis [5].

The purpose of this paper is to improve the methods previously used to show that the conditions (C_2) and (C_4)-(C_{11}) may be weakened to

$$L > \frac{\ln \lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2k\lambda_1}}{\lambda_1}, \tag{C_{12}}$$

where λ_1 is the smaller root of the equation $\lambda = e^{\kappa\lambda}$.

It is to be noted that as $k \to 0$, then all conditions (C_4)-(C_{11}) reduce to the condition (C_2), i.e.

$$L > 1.$$

However our condition (C_{12}) leads to

$$L > \sqrt{3} - 1 \approx 0.732$$

which is an essential improvement. Moreover (C_{12}) improves all the above conditions when $0 < k \leq \frac{1}{e}$ as well. For illustrative purpose, we give the values of the lower bound on L under these conditions when $k = \frac{1}{e}$:

(C_2):	1.000000000
(C_4):	0.966166179
(C_5):	0.892951367
(C_6):	0.863457014
(C_7):	0.845181878
(C_8):	0.735758882
(C_9):	0.709011646
(C_{10}):	0.599215896
(C_{11}):	0.471517764
(C_{12}):	0.459987065

We see that our condition (C_{12}) essentially improves all the known results in the literature.

2 Main Results

In what follows we will denote by k and L the lower and upper limits of the average $\int_{\tau(t)}^{t} p(s)ds$ as $t \to \infty$, respectively, i.e.

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds$$

and

$$L = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds.$$

Set

$$w(t) = \frac{x(\tau(t))}{x(t)}.$$

We begin with the preliminary analysis of asymptotic behavior of the function $w(t)$ for a possible nonoscillatory solution $x(t)$ of Eq. (1) in the case that $k \leq \frac{1}{e}$. For this purpose, assume that (1) has a solution $x(t)$ which is positive for all large t. Dividing first Eq. (1) by $x(t)$ and then integrating it from $\tau(t)$ to t leads to the integral equality

$$w(t) = \exp \int_{\tau(t)}^{t} p(s)w(s)ds \tag{3}$$

which holds for all sufficiently large t.

For the next lemmata see [12].

Lemma 1 *Suppose that $k > 0$ and Eq. (1) has an eventually positive solution $x(t)$. Then $k \leq 1/e$ and*

$$\lambda_1 \leq \liminf_{t\to\infty} w(t) \leq \lambda_2,$$

where λ_1 is the smaller and λ_2 the greater root of the equation $\lambda = e^{\kappa\lambda}$.

Lemma 2 *Let $0 < k \leq 1/e$ and let $x(t)$ be an eventually positive solution of Eq. (1). Assume that $\tau(t)$ is continuously differentiable and that there exists $\omega > 0$ such that*

$$p(\tau(t))\tau'(t) \geq \omega p(t) \tag{4}$$

eventually for all t. Then

$$\limsup_{t\to\infty} w(t) \leq \frac{2}{1 - k - \sqrt{(1-k)^2 - 4A}},$$

where A is given by

$$A = \frac{e^{\lambda_1 \omega k} - \lambda_1 \omega k - 1}{(\lambda_1 \omega)^2} \tag{5}$$

Remark 1 *It is easy to see that (4) implies that*

$$\int_{\tau(u)}^{\tau(t)} p(s)ds \geq \omega \int_u^t p(s)ds \quad \textit{for all } \tau(t) \leq u \leq t. \tag{4'}$$

Indeed, the function

$$v(u) = \int_{\tau(u)}^{\tau(t)} p(s)ds - \omega \int_u^t p(s)ds, \qquad \tau(t) \leq u \leq t,$$

satisfies the condition

$$v(t) = 0,$$

and

$$v'(u) = -p(\tau(u))\tau'(u) + \omega p(u) \leq 0.$$

If $p(t) > 0$ eventually for all t and

$$\liminf_{t\to\infty} \frac{p(\tau(t))\tau'(t)}{p(t)} = \omega_0 > 0,$$

then ω can be any number satisfying $0 < \omega < \omega_0$. Besides the case $p(t) \equiv p > 0$, $\tau(t) = t - \tau$ or the case $\tau(t) = t - \tau$ and $p(t)$ is $\tau-$periodic, there exists a class of functions which satisfy (4). Such a function is given in the Example below.

Lemma 3 *Let $0 < k \leq \frac{1}{e}$ and let $x(t)$ be an eventually positive solution of Eq.(1). Assume that condition (4) is satisfied. Then*

$$L \leq \frac{\ln \lambda_1}{\lambda_1} + \frac{-1 + \sqrt{1 + 2\omega - 2\omega\lambda_1 M}}{\omega\lambda_1}, \tag{6}$$

where λ_1 is the smaller root of the equation $\lambda = e^{k\lambda}$ and $M = \liminf\limits_{t\to\infty} \frac{x(t)}{x(\tau(t))}$.

Proof. Let θ be any number in $(1/\lambda_1, 1)$. From Lemma 1 and the definition of M, there is a $T_1 > T$ such that

$$\frac{x(\tau(t))}{x(t)} > \theta\lambda_1, \quad t \geq T_1, \tag{7}$$

and

$$\frac{x(t)}{x(\tau(t))} > \theta M, \quad t \geq T_1. \tag{8}$$

Now let $t \geq T_1$. Since the function $g(s) = x(\tau(t))/x(s)$ is continuous, $g(\tau(t)) = 1 < \theta\lambda_1$, and $g(t) > \theta\lambda_1$, there is $t^* \equiv t^*(t) \in (\tau(t), t)$ such that

$$\frac{x(\tau(t))}{x(t^*)} = \theta\lambda_1. \tag{9}$$

Dividing (1) by $x(t)$, integrating from $\tau(t)$ to t^* and taking into account (7), yields

$$\int_{\tau(t)}^{t^*} p(s)ds \le -\frac{1}{\theta\lambda_1}\int_{\tau(t)}^{t^*} \frac{x'(s)}{x(s)}ds = \frac{\ln(\theta\lambda_1)}{\theta\lambda_1} \tag{10}$$

Next we try to find an analogous inequality for

$$\Lambda := \int_{t^*}^{t} p(s)ds.$$

Integrating (1) from $\tau(s)$ to $\tau(t)$, we have

$$x(\tau(s)) - x(\tau(t)) = \int_{\tau(s)}^{\tau(t)} p(u)x(\tau(u))du, \qquad t^* \le s \le t.$$

Thus, integrating (1) from t^* to t and using (4′), we obtain

$$x(t^*) - x(t) = \int_{t^*}^{t} p(s)x(\tau(s))ds =$$

$$= \int_{t^*}^{t} p(s)[x(\tau(t)) + \int_{\tau(s)}^{\tau(t)} p(u)x(\tau(u))du]ds$$

$$\ge x(\tau(t))\int_{t^*}^{t} p(s)ds + x(\tau^2(t))[\int_{t^*}^{t} p(s)(\int_{\tau(s)}^{\tau(t)} p(u)du)ds]$$

$$\ge x(\tau(t))\int_{t^*}^{t} p(s)ds + \omega x(\tau^2(t))\int_{t^*}^{t} p(s)(\int_{s}^{t} p(u)du)ds$$

$$= x(\tau(t))\int_{t^*}^{t} p(s)ds + \frac{\omega}{2}x(\tau^2(t))(\int_{t^*}^{t} p(s)ds)^2$$

$$= \Lambda x(\tau(t)) + \frac{\omega}{2}\Lambda^2 x(\tau^2(t)),$$

where $\tau^2(t) \equiv \tau(\tau(t))$. Therefore

$$\Lambda + \frac{\Lambda^2}{2}\omega\frac{x(\tau^2(t))}{x(\tau(t))} \le \frac{x(t^*)}{x(\tau(t))} - \frac{x(t)}{x(\tau(t))}$$

(and taking into account (9) and (8))

$$\le \frac{1}{\theta\lambda_1} - \theta M.$$

Since, by (7),

$$\frac{x(\tau^2(t))}{x(\tau(t))} > \theta\lambda_1,$$

we obtain

$$\Lambda + \frac{\Lambda^2}{2}\omega\theta\lambda_1 \leq \frac{1}{\theta\lambda_1} - \theta M$$

or

$$\Lambda^2\frac{\omega\theta\lambda_1}{2} + \Lambda + (\theta M - \frac{1}{\theta\lambda_1}) \leq 0,$$

which leads to

$$\Lambda \leq \frac{-1+\sqrt{1-2\omega\theta\lambda_1(\theta M - \frac{1}{\theta\lambda_1})}}{\omega\theta\lambda_1} = \frac{-1+\sqrt{1+2\omega-2\omega\theta^2\lambda_1 M}}{\omega\theta\lambda_1},$$

since the other root is negative. Adding (10) and the last inequality, we obtain

$$\int_{\tau(t)}^{t} p(s)ds \leq \frac{\ln(\theta\lambda_1)}{\theta\lambda_1} + \frac{-1+\sqrt{1+2\omega-2\omega\theta^2\lambda_1 M}}{\omega\theta\lambda_1}.$$

Letting $\theta \to 1$ completes the proof. ■

Theorem. *Consider the differential equation (1) and let $L < 1$, $0 < k \leq \frac{1}{e}$ and there exists $\omega > 0$ such that (4) be satisfied. Assume that*

$$L > \frac{\ln\lambda_1}{\lambda_1} + \frac{-1+\sqrt{1+2\omega-2\omega\lambda_1 B}}{\omega\lambda_1}, \tag{11}$$

where λ_1 is the smaller root of the equation $\lambda = e^{k\lambda}$ and

$$B = \frac{1-k-\sqrt{(1-k)^2-4A}}{2}$$

where A is given by (5). *Then all solutions of Eq. (1) oscillate.* ■

Proof. Assume, for the sake of contradiction, that $x(t)$ is an eventually positive solution of Eq. (1). Then, by Lemma 3, we obtain (6) which, in view of Lemma 2, contradicts (11). The proof is complete. ■

Remark 2 *It is clear that in the above Theorem ω can be replaced by ω_0, where ω_0 is given in Remark 1.*

Remark 3 *Observe that when $\omega = 1$, then (11) reduces to*

$$L > \frac{\ln\lambda_1 - 1 + \sqrt{5-2\lambda_1+2k\lambda_1}}{\lambda_1}, \tag{12}$$

since from [12] it follows that

$$B = 1 - k - \frac{1}{\lambda_1}.$$

In the case that $k = \frac{1}{e}$, then $\lambda_1 = e$ and (12) leads to

$$L > \frac{\sqrt{7-2e}}{e} \approx 0.459987065.$$

Example. Consider the delay differential equation

$$x'(t) + px(t - a\sin^2\sqrt{t} - \frac{1}{pe}) = 0, \tag{13}$$

where $p > 0$, $a > 0$ and $pa = 0.46 - \frac{1}{e}$. Then

$$k = \lim_{t\to\infty} \inf \int_{\tau(t)}^{t} pds = \lim_{t\to\infty} \inf p(a\sin^2\sqrt{t} + \frac{1}{pe}) = \frac{1}{e}$$

and

$$L = \lim_{t\to\infty} \sup \int_{\tau(t)}^{t} pds = \lim_{t\to\infty} \sup p(a\sin^2\sqrt{t} + \frac{1}{pe}) = pa + \frac{1}{e} = 0.46.$$

Thus, according to Remark 2, all solutions of Eq. (13) oscillate. Observe that none of the results mentioned in the introduction apply to this equation. ■

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A VERSION OF GT GRAMMAR FOR MODERN GREEK LANGUAGE PROCESSING
PROPOSED AS A MODERN GREEK LANGUAGE CALL METHOD DEVELOPER

S.D. Baldzis S.A. Kolalas Ch.N. Savranidis

Abstract

In this paper, we put forward a version of a Generative Transformational Grammar (GTG), for the Modern Greek Language (MGL) Processing, which is used for the development of a Computer-Assisted Modern Greek Language Learning (CAMGLL) Method. This suggested Grammar, composes of the Template Grammars (TG's-generative dimension), the Basic Modern Greek Computational Multilexicon (BMGMLx) with its algorithms (transformational dimension). The model of this suggested Grammar is based on the structure and function of the MGL System, that is, MGL components, relations and functions of which it comprises. Thus, the teaching of the Grammar Code of the MGL is approached by the modern linguistic and communicative perception (Holistic Approach) with which morphology and syntax, forms and functions are inseparable. Furthermore, the suggested Grammar is able to parse and generate Modern Greek Sentences[1], in the framework of an Open Educational Environment where learning is experimental, creative and cooperative. The contents of the computational lexicons of the BMGMLx and the production rules of the TG's are suitably selected and enriched in order to use, firstly, words of themes and meanings from communicative areas, secondly, their dominant semantic combinations and thirdly, the commonly used morphological and syntactical rules. All these contents are functional for the basic forms of the communicative written MGL (Communicative Language Teaching Method). The Computer-Assisted Modern Greek Language Learning method based on the suggested Grammar can be used either in a classroom at school or by Internet correspondence, for teaching MGL as a native or foreign language.

1. Introduction

The reason for analyzing a Modern Greek Sentence, as it occurs in a sentence in every natural language, is understanding its contents, which means, identifying various actions, as well as the attributes that characterize the agents, the actions, and the recipients of these actions, for further use, for example in language teaching. Thus Modern Greek text processing presupposes: (a) the formalization of the MGL data, which are the vocabulary of the language, the syntax rules, the morphology rules and the semantic rules, i.e., the components, the relations and the functions that compose the structure and function of the MGL System, (b) Modern Greek text syntactic analysis or parsing, whereby each Modern Greek Sentence of

[1] When referring in this paper to the Modern Greek Sentences, we mean the whole (ακέραιες=akerees) and main (κύριες=kyries) sentences,

the text is "delinearized", i.e., a tree structure is extracted from the words which make up the sentence. This tree structure describes the role of each word in the Modern Greek Sentence. In parsing, the central role is played by the Grammar which is a device for giving the specifications of Modern Greek Sentences. However, the mechanisms of the structure and function of the MGL System lead to form this procedure for the MGL. In fact, the referred mechanisms form the required Grammar since it is a system which describes the ways of forming acceptable Modern Greek Sentences [1,4].

The MGL mechanisms are generative, that is, its application leads to the generation and the parsing of the syntactic structure of Modern Greek Sentences, which are classified[2] as simple, compound, amplified and compound-amplified [1,2,3,4,5]. These mechanisms are transformational whereby from every syntactic structure arises an infinite number of Modern Greek Sentences, using the appropriate sets of words and transformational rules each time. Applying the transformations to each word of the constructing sentence, the words form[3] the appropriate morphological type according to their syntactic role and comply with the rules of their semantic agreement. There are many and composite transformations as the MGL is an inflectional language. Similarly, in parsing a Modern Greek Sentence, the syntactic role of its words is found, i.e., Modern Greek Sentence syntactic structure.

In inflectional languages, the grammatical relations are expressed by declension, i.e., word suffixes, rather than the syntagmatic order of words or by the prepositions, as it occurs in the non-inflectional or in the semi-inflectional languages, like the English Language (EL) [1,4,21]. For example , in the EL a sentence which consists of a subject, a verb and an object can be expressed correctly, without having its meaning changed, only according to the pattern SVO; however in MGL, this sentence can be expressed correctly, having its meaning unchanged, according to the following six patterns: SVO, VSO, OVS, SOV, VOS and OSV. Thus the basic feature of MGL is that the syntactic relations are indicated by the case (see footnote 3). However, the semantic agreements which are held between the lexical items correspond to the semantic relations held between Subject-Verb, Verb-Object, and so on, in a sentence [1,4,5]. Hence, the semantic relations in MGL indicate the interdependence of syntax, semantics and morphology. Moreover, we note that specific word class or word classes are provided for each syntactic role.

Many schemes have been proposed for the natural language processing, which displayed certain inadequacies. It should be noted that such inadequacies were observed and

[2] As provided by the Modern Greek Syntax as it is taught at the Secondary Education Level .
[3] This in the Modern Greek Syntax, is expressed as follows: "the words irrespective of order found in a sentence and according to their syntactic role form their appropriate morphological type"; evidently complying with the rules of their semantic agreement.

recorded primarily in the case of the EL, which differs considerably from MGL, as mentioned above [14,15,16,17,20,21].

In recent years, a certain amount of research has appeared in the MGL processing. This research is usually orientated towards certain ranges of MGL processing [13,18,19,22, 23,24,26].

A method which addresses the full range of MGL processing, from the lexical level to the semantic one is a version of GTG's, whose model is composed of the Template Grammars (TG's-generative dimension), the Basic Modern Greek Computational Multilexicon (BMGMLx) with its algorithms (transformational dimension) [1,2,3,4,5,6,7].

Template Grammar is a modified version of the Grammar of Chomsky's hierarchy. This modification requires that by definition the production rules, which are finally syntactic structure rules, be grouped to templates of rules, producing the syntactic structure of the Modern Greek Sentences. The production rules are internationally established as phrase structure rules, mainly used for the English language processing. The syntactic categories of the inflectional MGL designate grammatical functions, which are inherent relation notions, rather than grammatical categories and express the interdependence of syntax, morphology and semantics [4,5]. In addition, results in the production and parsing an intermediate, abstract language of syntactical categories of MGL. This language is free of morphological rules and meanings having the same structure as Modern Greek Sentences and defines a Pattern Language, that is the Language of Syntactic Categories of MGL. Pattern Language is a formal language simply processed. The Template Grammars take on a special importance as an efficient tool for parsing and generating Pattern Language Sentences with the introduction of characteristic exponents. The characteristic exponents characterize the structure of the Pattern Language Sentences and map them to the corresponding templates of grammar rules which generate them. The characteristic exponents are strings of integers which are easily extracted and recognized. The characteristic exponents allow to find directly the template of generation of the Pattern Language Sentence which is proposed and hence the category of sentences where the individual sentence belongs avoiding the time consuming searching methods [1,2,3,4].

The Basic Modern Greek Computational Multilexicon (BMGMLx) is a system of computational and interconnecting lexicons which consists of recorded data concerning the vocabulary, the syntax, the morphology and the semantics of MGL. The BMGMLx for every word it contains and by means of its algorithms, can recognize or give any information about the word morphology and the word semantic agreement with other words in accordance with its syntactic role each time. This information is utilized by the algorithms of BMGMLx which describe "what has to be done" so that: (a) given the syntactic structure and chosen words to be transformed into an acceptable Modern Greek Sentence and (b) given a Modern Greek Sentence, the syntactic roles of its words and structure are pinpointed. In other words, the generative and transformational MGL mechanisms are finite rules which are expressed by algorithms. These algorithms recognize Modern Greek words, generate and parse their forms and their semantic combinations as well as Modern Greek Sentences [1,4].

According to the suggested Grammar, a Modern Greek Sentence is converted to the corresponding Pattern Language Sentence by means of the BMGMLx, its algorithms and Template Grammars (TG's). The resulting Pattern Language Sentence is expressed in the normal order of its syntactic categories as the Modern Greek Syntax defines; irrespective of its corresponding position of words in a given Modern Greek Sentence [1,4].

The proposed GTG as it is based on the structure and function of the MGL System continues the uniform perception of the GTG's of Noam Chomsky, where his model composes the morphological structure and the syntactic function of the language due to the semantic agreements of the words of the sentence [8]. Furthermore, the possibilities which derive from the realization and application of the suggested GTG in the teaching of the MGL guide the teaching of the Grammar Code of the MGL according to the modern linguistic and communicative perception. Based on these perceptions, morphology and syntax, form and function are inseparable; these principles are expressed by the Holistic Approach [12,28].

Moreover, the teaching of the morphology and syntax assisted by computer technology and the proposed Grammar is done without the pressure of rules and complex phrasing. Hence, the morphological and syntactic rules are presented in a simplified way through the emphasis on the production and parsing of commonly used Modern Greek Sentences of themes and meanings from communicative areas, in an Open Educational Environment. Thus the Communicative Language Teaching Method is followed. The BMGMLx with information it provides and is available at any time, makes the learning result more effective and efficient since it minimizes the metalanguage of morphological, syntactic and semantic rules of the MGL. In addition, the BMGMLx provides the necessary self-sufficiency for the user not to resort to other means of electronic or printed matter when finding language phenomena.

We note that, the proposed GTG is principled, computationally efficient, descriptively adequate for our purposes and introducing a MGL learning/teaching method development. Thus, the proposed GTG constitutes a Grammar Framework which allows for an accurate computational implementation and may form the syntactic component (Expert Module) of an Intelligent Computer-Assisted Modern Greek Language Learning (ICAMGLL) Method [9].

2. A Modified Version of GT Grammars for MGL Processing

2.1 Template Grammars and Characteristic Exponents – Generative Dimension

The application of Template Grammars in MGL processing led to the TG_1, TG_2, TG_3 και TG_4[4] with which the syntactic structures of the simple, the compound, the amplified and the compound-amplified sentences correspondingly are produced and parsed [1,2,3,4].

For example the TG_1 is defined as: $TG_1=(V_{M1},V_{T1},P_1,S)$, where: $V_{M1}=\{S,A,B,C\}$, $V_{T1}=\{a,b,c,g\}$, $P_1=\{p_{1,1}, p_{1,2}\}$, with $p_{1,1}=\{S\to AB, A\to a, B\to b\}$and $p_{1,2}=\{S\to AB, A\to a, B\to gC, C\to c\}$ [1,2,3,4].

We note that:

- S≡<<simple_sentence>>,A≡<<simple_subject>>,B≡<<simple_predicative>>,C≡<<simple _predicate>>, a≡<subject>, b≡<predicative_verb>, c≡<predicate>, g≡<conjunctive_verb>,
- $p_{1,1}$ and $p_{1,2}$ are the templates of production rules which produce the syntactic structures of the simple sentences,
- The production rules of $p_{1,1}$ and $p_{1,2}$ are the formalized syntactic rules, by means of the metalanguage BNF [1,2,3,]. Thus:

  <<sentence>>::=<<simple_sentence>>,
  <<simple_sentence>>::=<<simple_subject>><<simple_predicative>>,
  <<simple_subject>>::=<subject>,
  <<simple_predicative>>::=<predicative_verb> |
  <conjunctive_verb><<simple_ predicate>>,
  <<simple_predicate>>::=<predicate>.

The TG_1 produces the language $L(G_1\}=\{ab, agc\}$. The Pattern Language Sentences ab and agc are the patterns for an infinite number of Modern Greek Sentences.

Similarly TG_2, TG_3 and TG_4 are defined. We note that the templates of P_2 are twenty. However the templates of P_3 and P_4 are many because of the great variety of the MGL modifiers and their combinations. In our research, the commonly used types of amplified and compound-amplified sentences are included. These have a high rate of frequency in the Modern Greek texts. P_3 and P_4 are extended in the course of the research and the application of the proposed method [1,4].

The formalization of the syntactic structure rules of Modern Greek Sentences for processing led to [1,2,3,4]:

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[4] Although one TG could be defined describing all the Modern Greek Sentences syntactic structures, we introduce four distinct TG's each for every Modern Greek Sentences category so it achieves a small number of variables for each TG and greater transparency of the Pattern Language Sentences produced by the same template.

1. the substitution of the categories NP (Noun Phrase), VP (Verb Phrase), Art (Article), N (Noun), etc., of the phrase structure rules with the MGL syntactic categories <subject>, <predicate >, <predicative_verb>, <conjunctive_verb>, etc. of the MGL syntactic rules. The syntactic categories of the inflectional MGL simplify MGL processing since they are grammatical functions and express the interdependence of syntax, morphology and semantics. Thus, these functions in every syntactic category correspond to semantic[5] acceptable word in its acceptable morphological type and vice versa. On the contrary, the grammatical categories are insufficient for MGL processing since the syntactic roles of words which are provided depending on their order in a sentence do not consist in any case of sufficient condition detected in the syntactic roles in Modern Greek Sentences.

2. the generation and parsing of an intermediate and abstract Pattern Language, called Language of the Syntactic Categories. Pattern Language Sentences having the same structure as Modern Greek Sentences define Modern Greek Sentences deep structure, i.e. deep structure of Modern Greek Sentences is identical to the syntactic structure of Modern Greek Sentences and free of meanings.

3. two different Modern Greek Sentences that is two different surface structures as we define can have either the same deep structure, if having the same syntactic structure, or different deep structure, if having different syntactic structure. Pattern Language has formal language features and behavior which is easily programmed. These affect the acceleration of Modern Greek Sentences processing since they simplify the formalization, the generation and parsing of the Pattern Language Sentences, detecting directly the template of its production rules by means of the characteristic exponents. The characteristic exponents are strings of integers which are easily extracted and recognized. The characteristic exponents characterize the structure of the Pattern Language Sentences and map them to the corresponding templates of which generate them avoiding the time consuming searching methods.

For example (see Table 1) :

a) The characteristic exponents of the Pattern Language Sentence ab is the string κ,ρ_1,ρ_2 = 0,1,0, and the characteristic exponents of Pattern Language Sentence agc is the string $\kappa,\rho_1,\rho_2,\lambda$ = 0,0,1,0.

b) The template $p_{1,2}$ generates the unique Pattern Language Sentence agc.

c) The template $p_{2,18}$={S→AB, A→a|Ava|Aua, B→gc, C→Cuc}, is one of the templates which generates the syntactic structures of the compound sentences. $p_{2,18}$ generates the class

[5] The interdependence of syntax and semantics is indicated by the semantic relations between Subject-Verb, Verb-Object, and so on. We note that, similar to the semantic relations between the syntactic categories in Pattern Language Sentences (deep structure), the semantic agreement is defined between the lexical items in the corresponding Modern Greek Sentences (surface structure) [1,2,3,4,5,6,7].

of Pattern Language Sentences $a(v^i a)^\kappa u^1 a^1 gc(v^i c)^\lambda u^1 c^1$, where $i = 0,1$ and $\kappa, \lambda > o$, that is, $p_{2,18}$ generates more than one Pattern Language Sentence, e.g.,

for $\kappa,i,-,j,\kappa,\rho_1,\rho_2,\lambda,i,-,j,\lambda = 1,1,-,1,1,0,1,1,1,-,1,1$ the corresponding Pattern Language Sentence is avauagcvcuc[6],

for $\kappa,i,-,j,\kappa,\rho_1,\rho_2,\lambda,i,-,j,\lambda = 2,1,-,1,1,0,1,2,1,-,1,1$ the corresponding Pattern Language Sentence is avavauagcvcvcuc, and so on..

Templates of Syntactic Rules	Structures of Syntactic Categories and Classes of Structures of Syntactic Categories	Characteristic Exponents																				
		κ	i	j	j	κ	...	κ_2	i	j	j	κ_2	ρ_1	ρ_2	λ	i	j	j	λ	...	μ	...
$p_{1,1}$	ab	0											1	0								
$p_{1,2}$	agc	0											0	1	0							
$p_{2,18}$ *	$a(v^i a)^\kappa u^i a^i gc(v^i c)^\lambda u^i c^i$	>0	0|1	-	1	1							0	1	>0	0|1	-	1	1			
$p_{3,20}$ *	abe_1	0											1	0							1	
$p_{4,21}$ *	$ad_1(v^i ad_1)^\kappa u^i(ad_1)^j b$							>0	0|1	-	1	1	1	0								

Table 1: 1. $\rho_1 = 0|1$, 0 indicates that the verb is not a predicative verb b, while 1 indicates that the verb is a predicative verb b.

2. $\rho_2 = 0|1$, το 0 indicates that the verb is not a conjunctive verb g, while 1 indicates that the verb is a conjunctive verb g.
3. $\kappa = 0, 1, 2, 3, \ldots$ indicates that there is 1, 2, 3, . . . times the subject a.
4. $\kappa_2 = 0, 1, 2, 3, \ldots$ indicates that there is 1, 2, 3, . . . times the pair ad_1.
5. $\lambda = 0, 1, 2, 3, \ldots$ indicates that there is 1, 2, 3, . . . times the predicate c.
6. $\mu = 0, 1, 2, 3, \ldots$ indicates that there is 1, 2, 3, . . . times the object e_1.
7. i = 0|1, similarly if the punctuation "," i.e. v is omitted or not.
8. j = 0|1, similarly if the conjunctive, e.g. "and", i.e. u is omitted or not.
9. in case that one of i or j doesn't belong to the syntactic structure the dash "-" is corresponded to the CE i or j.

* The corresponding templates in details are:

$p_{2,18} = \{ S \to AB, A \to a|Ava|Aua, B \to gc, C\ Cuc\}$,

$p_{3,20} = \{ S \to AB, A \to a, B \to bE_1, E_1 \to e_1 \}$,

$p_{4,21} = \{ S \to AB, A \to aD_1|AD_1vaD_1|AD_1uaD_1, B \to b, D_1 \to d_1 \}$.

It is proven that in each template corresponds to a unique combination of characteristic exponents. Also Pattern Language Sentences which belong to the same class of syntactic structures correspond to unique values of the unique combination of characteristic exponents [1,2,3,4].

2.2 Basic Modern Greek Multilexicon and its Algorithms -Transformational Dimension

In the proposed model in order to transform the abstract Modern Greek Sentences deep structure to surface structures, i.e., Modern Greek Sentences, we insert meanings into the deep structures, following prescribed rules [1,2,3,4,5]. That is, based on semantic specifications, we attach words to the syntactic categories of a Pattern Language Sentence, converting it in the beginning into a sentence form with meanings (transformation level of semantic synthesis).

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[6] Η Ειρήνη, η Σοφία και ο Δημήτρης είναι φρόνιμοι, επιμελείς και ευγενικοί

I Irini, i Sofia ke o Dimitris ine fronimi epimelis ke evgeniki

Irini, Sofia and Dimitris are sensible, diligent and polite

is one of the infinite Modern Greek Sentences (surface structure) which corresponds to the Pattern Language Senetence or deep structure avauagcvcuc.

Then according to the Modern Greek morphological rules, the appropriate types of words in the sentence form with meanings are formed (transformation level of morphological formation). Thus, the surface structure or the acceptable Modern Greek Sentence results. Similarly Modern Greek Sentences are transformed to their corresponding deep structures.

The information which is used in the two transformation levels (obligatory transformations) is the information which every word is given in Modern Greek Sentences; it is morphological, syntactical information of the semantically accepted matching, depending each time on the syntactic roles of the words in a sentence. This information which is formalized, codified and filed define the Basic Modern Greek Computational Multilexicon (BMGMLx) which is used by its algorithms, as they describe "what should be done" in order to transform a deep structure sentence into a surface structure and vice versa detecting the syntactic roles of the words in Modern Greek Sentence.

The BMGMLx is a system of computational and interconnecting lexicons made up of four - unit - lexicons of MGL. Every unit-lexicon of this system corresponds to one of the four different dimensions of the information content, thus [1,4,5]:

1. The Modern Greek Computational Lexicon (MGCLx) contains, in alphabetical order, all the words that the proposed system recognizes and processes. The contents of MGCLx is enriched in the course of the research and the application of the proposed method.

2. The Logical Computational Lexicon of Basic Meanings (LCLxBM) and the Computational Lexicon of Context of MGL (CLxCMGL): The items of these lexicons are entries of the Modern Greek Computational Lexicon, whereby in the former they are ordered in a strict succession of meanings of its contents and in the latter are formed in semantic rules. LCLxBM gives the synonyms and antonyms of words, phrases and idioms of the words of Modern Greek Computational Lexicon. CLxCMGL produces and recognizes the dominant semantic combinations of its items according to their semantic relations and agreement.

3. The Modern Greek Morphology Computational Lexicon (MGMCLx) contains all kinds of morphological information which every word of Modern Greek Computational Lexicon may hold in order to formulate an accepted morphological type of word or to detect its type.

4. The Modern Greek Syntax Computational Lexicon (MGSCLx), are the prescriptions of Modern Greek Syntax which provide the agreement of the morphological types of the sentence terms according to their syntactic role in the sentence.

The morphological formation is the most mechanical phase of the obligatory transformations, requiring knowledge of Modern Greek Morphology, appropriate recording of data as well as appropriate codification of its mechanisms and mechanisms of its interdependencies of Modern Greek Syntax and the items of the Modern Greek Computational Lexicon [1,4].

On the contrary, the semantic synthesis is the most difficult phase of the transformation procedure, mainly concerning the definition, the designing and the formalization of the semantic rules [5]. The required semantic information is not recorded in a similar way as the syntactic information, with which the Modern Greek Sentence deep structures are described. Thus a procedure similar to that of syntax is proposed to be formulated, whereby from a finite number of dominant semantic rules or semantic criteria of matching words, a direct semantic manipulation of an infinite set of words occurs, that is, the generation or parsing an infinite number of Modern Greek Sentences. Simultaneously, it provides the ability from the same procedure to minimize the possibility of non-manipulation of ambiguous meanings, conversely maximizing the possibility of recording and controlling the possible meanings of words or idioms or common phrases, their synonyms and antonyms. Generally speaking, the possibility of usage of lexicon provides "the entire lexicon" and its direct further enrichments.

1. GENERAL		2. SPECIFIC	
1.ABSTRACT	2.CONCRETE	3.MATERIAL	4.IMMATERIAL
1.ΥΠΑΡΞΗ (HYPARXI) EXISTENCE 2. ΣΧΕΣΗ (SCHESI) RELATION 3. ΕΝΟΤΗΤΑ (ENOTHTA) UNIT 4.ΤΑΞΗ (TAXI) ORDER 5.ΠΟΣΟΤΗΤΑ (POSOTITA) QUANTITA 6.ΑΡΙΘΜΟΣ (ARITHMOS) NUMBER 7.ΧΡΟΝΟΣ (CHRONOS) TIME	8.ΧΩΡΟΣ (CHOROS) SPACE 9.ΔΙΑΣΤΑΣΗ (DIASTASI) DIMENSION 10. ΣΧΗΜΑ (SCHIMA) FORM 11.ΕΝΕΡΓΕΙΑ (ENERGIA) ENERGY 12.ΚΙΝΗΣΗ (KINISI) MOTION	13.ΑΝΟΡΓΑΝΗ ΥΛΗ (ANORGANI HYLI) INORGANIC MATERIAL 14.ΟΡΓΑΝΙΚΗ ΥΛΗ (ORGANIKI HYLI) ORGANIC MATERIAL	15.ΝΟΥΣ (NOUS) MIND 16.ΒΟΥΛΗΣΗ (VOULISI) MIND 17.ΔΡΑΣΗ (DRASI) ACTION 18.ΑΞΙΕΣ (AXIES) VALUES 19.ΣΥΝΑΙΣΘΗΜΑ (SYNESTHIMA) FEELING 20.ΗΘΟΣ (ITHOS) ETHOS 21.ΘΕΟΣ (THEOS) GOD

Table 2: The 21 Broader Basic Units of Meanings-Chapters (BBUM-Ch's) which are classified into 4 sub-categories of BBUM-Ch are also divided into 2 categories of BBUM-Ch.

Thus, to formalize the dominant semantic rules, we adopt the model of the Logical Lexicon [5]. The material of the Logical Computational Lexicon of Basic Meanings (LCLxBM) are the items of the Modern Greek Computational Lexicon (MGCLx) which are codified with a strict succession of the general meanings and grouped to their partial meanings including the synonyms, related words and language expressions. Every word belongs to one of the 21 Broader Basic Units of Meanings-Chapters (BBUM-Ch's) of Table 2.

Every Broader Basic Units of Meanings-Chapter (BBUM-Ch) includes its general meanings by which are named Meanings-Chapters (M-Ch's). For example, the BBUM-Ch "Υπαρξη" (Hyparxi=Existence) contains 7 M-Ch's of Table 3.

1.Υπαρξη (Hyparxi) Existence	2.Ανυπαρξία (Anhyparxia) Non-Existence
3.Κατάσταση (Katastasi) Situation	
4.Περίσταση (Peristasi) Occasion	
5.Εσωτερικός Κόσμος (Esoterikos Kosmos) Inner World	6. Εξωτερικός Κόσμος (Exoterikos Kosmos) Outer World
7.Εγώ (Ego) Ego	

Table 3: The BBUM-Ch "1_ΥΠΑΡΞΗ" (HYPARXI=EXISTENCE) contains 7 M-Ch's.

For every one of its 1500 Meanings-Chapters (M-Ch's), the words, idioms or common phrases, which by meaning belong to the same general meaning, are included, and are classified in corresponding paragraphs. That is, every M-Ch consists of as many paragraphs as there is of inflected and uninflected words, idioms or common phrases which appear in the same general meaning. For example the M-Ch "Υπαρξη" (Hyparxi=Existence) contains only the 5 paragraphs of Table 4.

p.o	Paragraphs	Words belongs to BMGLx
1	κ1_noun	ύπαρξη (hyparxi=existence), υπόσταση (hypostasi= subsistence), το είναι (to ine=being), etc.
2	κ3_intransitive_verb	υπάρχω (hyparcho=exist), είμαι (ime=be), υφίσταμαι (hyfistame=exist), etc.
3	κ4_transitive_verb	δίδω υπόσταση (dido hypostasi=lead to existence), διατηρώ (diatiro=maintain), etc.
4	κ5_impersonal_verb	υπάρχει (hyparchi=there is), είναι (ine=it is), βρίσκεται (vriskete=it is found), etc.
5	κ6_adjective	υπαρκτός (hyparktos=existent), υπάρχων (hyparchon=existent), υφιστάμενος (hyfistamenos=being), etc.

Table 4: The M-Ch "1_Υπαρξη" (Hyparxi=Existence) contains only 5 paragraphs, where p.o means paragraph order.

In every paragraph, words are grouped according to the partial meanings where-by they are recorded and cited directly after the corresponding synonyms and related words. For example, we give some of these groupings according to the partial meanings of the words where the paragraph "noun" of the M-Ch "Υπαρξη" (Hyparxi=Existence) contains the 11 groups of Table 5

pmw.o	Words / nouns belongs to BMGLx
1.	ύπαρξη (hyparxi=existence), υπόσταση (hypostasi=subsistence), (το) είναι (to ine=being), οντότητα (ondotita=entity),
2.	αυθυπαρξία (afthyparxia=self-existence), αυτοτέλεια (aftotelia=self-sufficient), αυτοζωή (aftozoi=self-being) , αυτοζωία (aftozoia=self-existence),
3.	προΰπαρξη (prohyparxi=pre-existence), προϋπόσταση (prohypostasi=pre-subsistence),
4.	συνύπαρξη (synhyparxi=coexistence),
5.	ενύπαρξη (enhyparxi=in-existence),
6.	διατήρηση (diatirisi=preservation), παραμονή (paramoni=stay),
7.	διάσωση (diasosi=salvage), περίσωση (perisosi=save), επιβίωση (epiviosi=survival),
8.	όν (on=being), οντότητα (ondotita=entity), πλάσμα (plasma=creature),
9.	άτομο (atomo=individual), πρόσωπο (prosopo=person), ψυχή (psyhi=soul), κανείς (kanis=anyone), κανένα (kanena=anybody), τις (tis=somebody), ένας (enas=someone), κάποιος (kapios=some),
10.	αντικείμενο (andikimeno=object), πράγμα (pragma=thing), κάτι (kati=something), τίποτε (tipote=anything),
11.	οντολογία (ondologia=ontology).

Table 5: The 11 groups according to the partial meanings of the words (pmw) which the paragraph "1.noun" of the M-Ch "1_Υπαρξη" (Hyparxi=Existence) contains, where pmw.o means partial meaning word order.

For instance the verb "μελετώ"[7] (meleto=study) is the first verb of paragraph "κ2.1", i,e, "2.verb(transitive-intransitive)" of the 828_M-Ch "Μάθηση" (Mathisi=Learning) of 15_BBUM-Ch "Νούς" (Nous=Mind). This verb and all its synonyms and its relations, which belong to the same paragraph, are accepted semantically as subject of any word of paragraph "κ.1" i,e, "1.noun" belongs to one or more M-Ch's having further the semantic feature "ανθρώπινο_όν" (anthropino_on=human_being). We define as semantic category of the MGL all the words that belong to the same paragraph of one or more M-Ch's and, in turn, each time can substitute a specific syntactic category with a specific combination of words in a sentence form with meanings [1,4,5].

We note that the semantic categories may be defined, apart from the grouping of words of the same paragraph, also by groupings of semantic categories always being based on

[7] this verb has multiple meanings, this example refers to one of these meanings, similarly this applies to the other meanings as well.

a common semantic feature which acts as a semantic prescription of substitution of the specific syntactic category from the pool of words of the paragraph which they belong to. These groupings concern all types of paragraphs of the M-Ch [1,4,5].

The semantic category is represented by the unaccentuated Greek word, which specifies the common feature that groups words, in the pair <...>. There are no specific rules but only principles that lead to the definition of semantic categories. A sample of noun paragraph grouping belonging to a multiple M-Ch, as shown in the grouping of Table 6, shows the way this procedure is defined [1,4,5].

<table>
<tr><th>s.c.o</th><th colspan="3">Semantic Categories</th></tr>
<tr><td>1.</td><td><εμψυχ_ον>
<empsychon>
<animate></td><td>::=</td><td><ανθρωπιν_ον> |<ζω_ον> |<πετειν_ον> |<ψαρι_ον> |<εντομ_ον> |<ερπετ_ον>,
<anthropin_on> <zo_on> <petin_on> <psari_on> <entom_on> <erpet_on>
<human> |<animal> |<bird> |<fish> |<insect> |<reptile>,</td></tr>
<tr><td>2.</td><td><ανθρωπιν_ον>
<anthropin_on>
<human></td><td>::=</td><td><ανθρωπος> | ... | <διδασκαλος> | ... | <πολεμιστης> | ... | <συγγραφεας> | ... ,
<anthropos> <didaskalos> <polemistis> <sygrafeas>
<man> | ... | <teacher> | ... | <warrior> | ... | <writer> | ... ,</td></tr>
<tr><td>3.</td><td><ανθρωπος>
<anthropos>
<man></td><td>::=</td><td>άνθρωπος | απόγονος του Αδάμ | λογικό όν | ανθρωπάκι | ... ,
anthropos apogonos tou Adam logiko on anthropaki
man | descendant of Adam | logical being | little man | ... ,</td></tr>
<tr><td>4.</td><td><διδασκαλος>
<didaskalos>
<teacher></td><td>::=</td><td>διδάσκαλος | δάσκαλος | νηπιαγωγός | ... | καθηγητές | ... ,
didaskalos daskalos nipiagogos kathigitis
teacher | school master | kindergarten teacher | ... | professor | ... ,</td></tr>
<tr><td>5.</td><td><συγγραφεας>
<syngrafeas>
<writer></td><td>::=</td><td>συγγραφέας | διηγηματογράφος | ... | δημοσιογράφος | ... ,
syngrafeas diigimatografos dimosiografos
writer | writer of short stories | ... | journalist | ... ,</td></tr>
<tr><td>6.</td><td><πολεμιστης>
<polemistis>
<warrior></td><td>::=</td><td>πολεμιστής | αγωνιστής | στρατιωτης | πεζοναυτης | ... .
polemistis agonistis stratiotis pejonaftis
warrior | fighter | soldier | marine | ... .</td></tr>
</table>

Table 6: A sample of semantic category definition, where s.c.o means semantic category order.

Semantic rules are the acceptable and the dominant semantic combinations between the words of a paragraph or the words of the same type of paragraphs of different M-Ch's with words of other similar types of paragraphs of different M-Ch. This is achieved by the dominant semantic combinations which are based on the meanings of the words as well as the semantic relations between the words as required by the corresponding syntactic roles. The semantic rules are procedures which are described by means of the syntactic categories [1,4,5]. For example, the expression:

υποκειμενο[8](μελετώ)::=<ανθρωπιν_ον> or

[8] i.e., subject.

υποκειμενο(15.0828.κ2.1)::=14.0543.κ1.1|15.0831.κ1.2|17.1111.κ1.3|15.0956.κ1.1|..., (see Tables 2-7, screen 1) comprises a semantic rule which defines the subjects of the verb"μελετώ" (meleto=study).

MGLx	LCLxBM BBUM-Ch . M-Ch . Paragraph . w.o	MGMLx	CLxCMGL
w.o. άνθρωπος	ORGANIC MATERIAL. Man . noun . w.o 14 . 0543 . κ1.1 . w.o	MN.003	<ανθρωπος> <man>
w.o. δάσκαλος	MIND . Teacher. noun . w.o 15 . 0831. κ1.2. w.o	MN.003	<δι∂ασκαλος> <teacher>
w.o. μελετάω-ώ	MIND. Learning . verb . w.o 15 . 0828 . κ2 . w.o MIND . Examination . verb . w.o 15 . 0845 . κ3 . w.o MIND . Memory . verb . w.o 15 . 0918 . κ3 . w.o WILL . Purpose . verb . w.o 16. 1010 . κ3 . w.o	RM.105	<ανθρωπομαθηση> <human_learning> <παρατηρηση_εξεταση> <examination_observation> <ανθρωπομνεια> <human_memory> <σκοπος> <purpose>
w.o. στρατιώτης	ACTION . Warrior . noun . w.o 17 . 1111 . κ1.3. w.o	MN.017	<πολεμιστης> <warrior>
w.o. συγγραφέας	MIND . Writer. noun . w.o 15. 0956 .κ1.1. w.o	MN.056	<συγγραφεας> <writer>

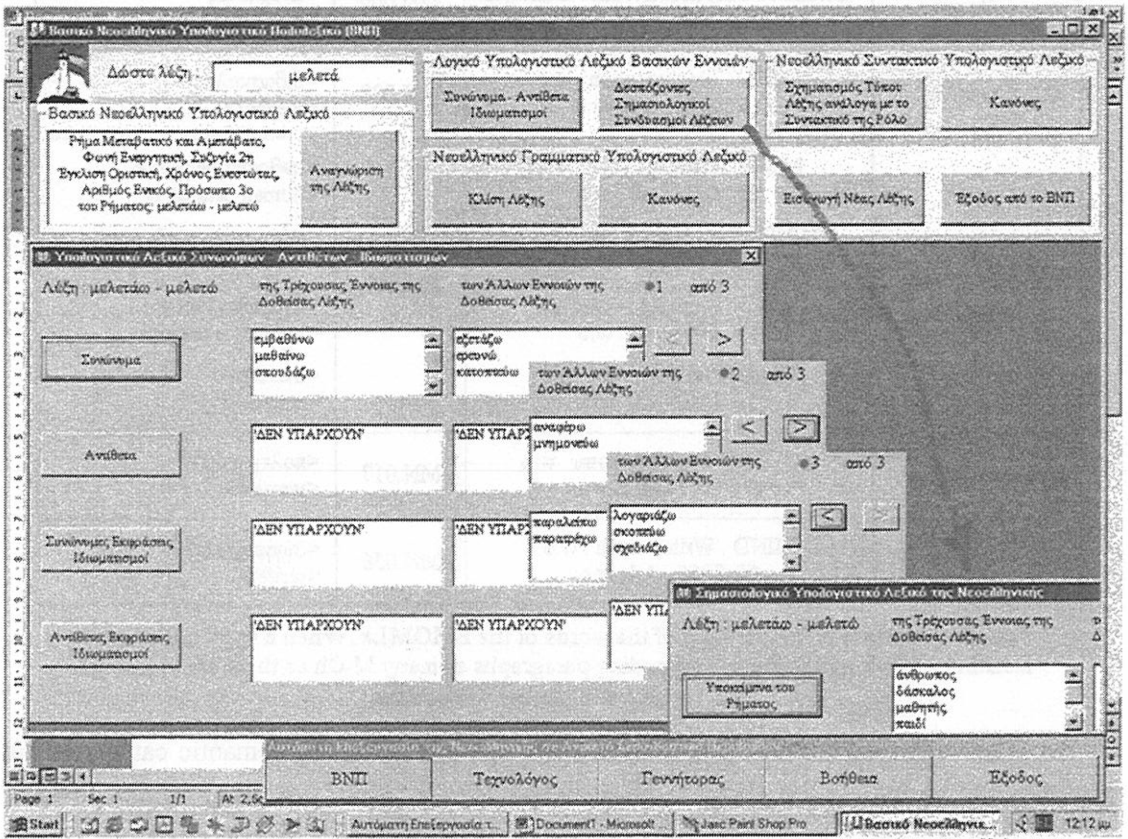
Table 7: A sample of codification of the words of the BMGMLx. When a word has multiple meanings, it belongs to the corresponding paragraphs as many M-Ch as there are meanings, where w.o. means word order.

The variety of the semantic rules or the combinations of the semantic categories are just as many as there are the variety of the syntactic categories and the semantic relations between them, as seen in the Modern Greek Syntax. Moreover, with a finite number of semantic rules, an infinite number of lexicon entries may be combined defining the semantic structures. The set of the MGL semantic rules, the set of the MGL semantic categories and the MGL semantic structures constitute the semantic basis of MGL [1,4,5].

3. The Proposed Modern Greek Sentences Generators and Parsers the Key-stones of a CAMGLL Method Development

It can be taken for granted that a Natural Language does not simply comprise a set of words expressing simple meanings, but a set of words put into use communicatively and interrelated with morphological rules and syntactic structures in speech. Therefore, composite meanings and concepts can be expressed. For the user, the aim of a Computer-Assisted Modern Greek Language Learning (CAMGLL) method without using the overload of rules and extensive phrasing can be made aware of the mechanisms of the language (profound knowledge) and the acquisition of the ability to produce, comprehend and process written and spoken texts (ability of use). The degree by which a CAMGLL method can provide efficiency with the response to the pursuit of a CAMGLL aim mainly depends on the efficiency of its design [6,7,9,25,27], (see Screens 1,2).

Thus, the efficiency of the proposed Grammar for the development of the CAMGLL method can be shown by the steps of parsing the Modern Greek Sentences concerning the processing of components, relations and functions of the MGL System.



Screen 1: The computational lexicons of the BMGMLx interconnected with Microsoft Programs or not, have the efficiency to recognize and give information of morphological, syntactic and the semantic combination in general or specific for every word they contain, if the word belongs to a particular sentence

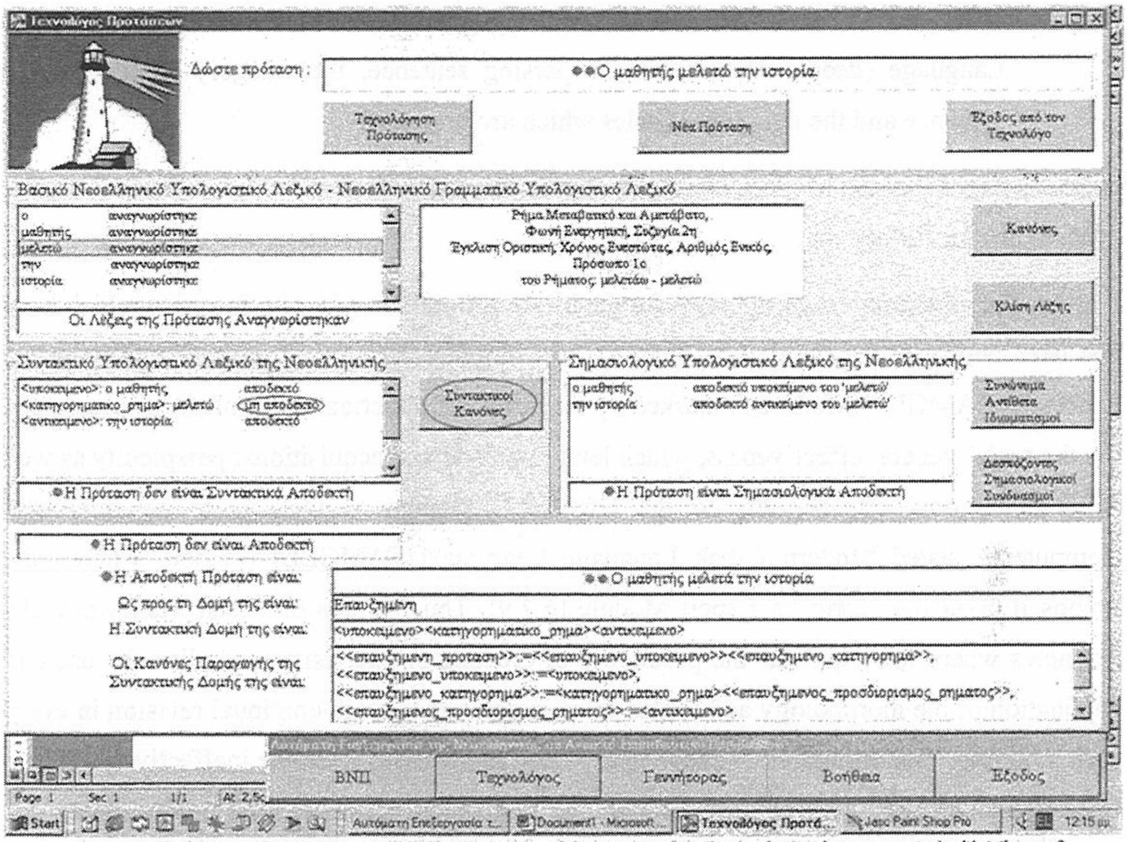
Step 1 The keying of the parsing sentence (surface structure)

Step 2 Lexical and Morphological recognition of the words of the sentence by means of:

1. the Modern Greek Computational Lexicon (MGCLx) and
2. the Modern Greek Morphology Computational Lexicon (MGMCLx)which are Unit-Lexicons of the Basic Modern Greek Computational Multilexicon (BMGMLx)

- Checks if the words of the parsing sentence belong to the MGCLx.
- Returns information for the part of speech for each word of the parsing sentence and information for its morphological type if flectional.
- Possibility of analytically showing the morphological types of any word in the parsing sentence.

- Possibility showing any Related Morphological Rule.
- Possibility of pinpointing a non-morphological type of any word of the parsing sentence and its error correction (corrector).
- Possibility of interconnections with commonly-used Microsoft Programs.



Screen 2: The Parser of MGMS's, as with the Generator, of the proposed method, whether interconnected with Microsoft programs or not, gives the pupil the opportunity of writing in MS Word for example the ability to parse any MGMS or its words. In the case of written mistakes of morphology, syntax or semantics, the system detects, gives the opportunity of explanation to the ignored rules and corrects those mistakes, as in the example of the screen above.

Step 3 Recognition of syntactical role of words of sentences by means of the Modern Greek Syntax Computational (MGSCLx).

- Possibility of pinpointing of non-morphological type relating to the syntactic role of any word of the parsing sentence (corrector)
- Possibility of showing any related syntactic rule.
- Possibility of interconnections with commonly-used Microsoft Programs.

Step 4 Recognition of semantic combination of words of the sentence by means of the The Logical Computational Lexicon of Basic Meanings (LCLxBM) and the Computational Lexicon of Context of MGL (CLxCMGL):

- Possibility of pinpointing unacceptable semantic combination relating to the syntactic role of any word of the parsing sentence.
- Substitution of any word in the parsing sentence with their synonyms and antonyms.

- Possibility of showing for any word of the parsing sentence, its possible semantic combination of words of the MGCLx in accordance with its syntactic role each time.
- Possibility of interconnections with commonly-used Microsoft Programs.

Step 5 Following Steps of 1,2,3 and 4 there is the appearance of the corresponding Pattern Language (deep structure) of the parsing sentence, the category of the parsing sentence and the template of rules which are produced.

It can be noted that the Modern Greek Sentences generators follow similar steps with similar efficiencies for further use.

The effectiveness of the tools for the Modern Greek Sentences processing which have just been described for the development of the Computer-Assisted Modern Greek Language Learning (CAMGLL) Method is marked by the structure, function and content of these tools, i.e. the tools' general effectiveness, which lend linguistic and acquisitional perspicuity as well as computational effectiveness to the CAMGLL Method as well as to an Intelligent Computer-Assisted Modern Greek Language Learning (ICAMGLL) Method, where the proposed GTG may form its Expert Module [6,7,9]. Thus, in this proposed framework, the examples where the pupil has the possibility of creating or exercising underline the use and application of the morphology and syntax rules in practice through continual revision in every unit; avoiding long, monotonous theoretical and in many cases tiresome ineffective phrasing. Cooperative learning is promoted through the realization of the Modern Greek Sentences processing in an open experimental and creative environment from the pupil for written Modern Greek Sentences by means of pedagogical methods [10,11,25]. Given simultaneous emphasis on the language practice which is an element of the modern language teaching [12,28]

Another example promoted by the proposed grammar is the independent use of the computational lexicons of the Basic Modern Greek Computational Multilexicon (BMGMLx) through which each word they contain are able to recognize and give information for its morphology, syntax and semantic combination in general or specific if the word belongs to a particular sentence. The BMGMLx with information it provides and is available at any time, makes the learning result more effective and efficient since it minimizes the metalanguage of morphological, syntactic and semantic rules of the MGL. In addition, the BMGMLx provides the necessary self-sufficiency for the user not to resort to other means of electronic or printed matter when finding language phenomena.

The realization of the CAMGLL Method works in the commonly used Windows'95, '98, 2000 and XP.

4. Conclusions

The goal of this study was to apply a version of a GTG for the MGL Processing, in order to use it in the development of a Computer-Assisted Modern Greek Language Learning (CAMGLL) Method. The suggested GTG is composed of the Template Grammars - generative dimension and the Basic Modern Greek Computational Multilexicon (BMGMLx) with its algorithms - transformational dimension.

Template Grammars are a version of GTG of the Chomsky hierarchy, with the additional property to group the production rules, which generate the strings of the corresponding language. Template Grammars generate the Pattern Language of MGL. Pattern Language Sentences are free of morphological forms and meanings, but with the same syntactic structure as the corresponding Modern Greek Sentences. Characteristic exponents, which are also introduced, characterize Pattern Language Sentences structure and automatically map them to the corresponding templates of rules, from which they were generated and vice versa, avoiding time consuming search methods.

Basic Modern Greek Computational Multilexicon (BMGMLx) is a system of computational and interconnecting lexicons which consists of recorded data concerning the vocabulary, the syntax, the morphology and the semantics of MGL. BMGMLx algorithms describe the finite rules, which express the generative and transformational MGL mechanisms and recognize Modern Greek words, generate and parse their forms and their semantic combinations as well as Modern Greek Sentences. For this, the Semantic Basis of MGL is also introduced.

This GTG model is based on the structure and function of the MGL System. Thus, the teaching of the Grammar Code of the MGL is based on the Holistic Approach. Furthermore, the suggested GTG is able to parse and generate Modern Greek Sentences, in the framework of an Open Educational Environment where learning is experimental, creative and cooperative.

The contents of the computational lexicons of the BMGMLx and the production rules of the Template Grammars are suitably selected and enriched in order to use, firstly, words of themes and meanings from communicative areas, secondly, their dominant semantic combinations and thirdly, the commonly used morphological and syntactical rules. All these contents, interconnected with Microsoft Programs or not, are functional for a Communicative MGL Teaching Method, based on the written Modern Greek Sentences, avoiding long, monotonous theoretical and in many cases tiresome ineffective phrasing. Also, BMGMLx provides the necessary self-sufficiency for the user not to resort to other means of electronic or printed matter when finding language phenomena.

The CAMGLL method based on the suggested GTG can be used either in a classroom at school or by Internet correspondence, for teaching MGL as a native or foreign language.

Moreover, the suggested GTG, on the one hand constitutes a Grammar Framework which may form the Expert module of an ICAMGLL Method, on the other hand describing the structure and function of other Natural Language Systems may introduce a CALL Method as well as the Expert modules of an ICALL Method, for the corresponding Languages.

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